12th IMECO TC1&TC7 Joint Symposium on
Man Science & Measurement
September, 3-5, 2008, Annecy, France

UNCERTAINTY MEASUREMENT AND DYNAMIC SYSTEM CHAOTICAL BEHAVIOUR

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Abstract Chaotic behaviour of nonlinear dynamic system as a mechanism of influence on uncertainty measurement are discussed. Based on the logistic equation as a mathematical model, measeurement uncertainty investigated. In the case when dynamic chaos take place in the system, uncertainty measurement has anormal distribution.

Keywords: uncertainty, dynamic chaos, logistic equation.

1. INTRODUCTION

The GUM are based on two physical models, which described the parameters of the examination objects or systems and its quantity random variable.

According to the first GUM model, measurands of static objects or dynamic system must be described by well-defined physical quantities, represented by the essentially unique values [1]. The second GUM model described random variable of the measurement quantity as a result of the influence in and out ergodic random processes on the examination objects. Physical models allow usage of well-known statistical methods for analysis of measurement results and estimation uncertainty measurement.

These two physical models ensure carry out all GUM requirements under the estimation physical quantities relevant only stable and stationary state of the examination system.

For dynamic systems (physical or chemical) with chaotic behavior (as example strange attractor [2]) estimation measurement results must to take into account the character of dynamical changes [3, 4].

Generally, the behavior of a dynamic system can be described with a set of low- dimensional nonlinear differential equations that may have two different types of solutions. A common type of mathematical system with non-regular solutions is a system of nonlinear discrete difference equations that are well-known in literature [5]. One type of solutions describes deterministic periodical behavior in time while other type of solutions describes chaotic or non-regular behavior. We can observe random or chaotic behavior in real systems that make influence on measurement results in these dynamical systems.

Since the GUM gives no recommendations how to estimate measurands and uncertainty measurements in the dynamic systems with a chaotic behavior.

This report present the investigations of mathematical model, which take into account deterministic and chaotic solutions, on conditions of estimations uncertainty measurement.

2. PURPOSE

2.1. Classic determinism in the theory of measuring

The analysis of the determined processes is built on the basic postulates of determinism classic physics. In accordance with one of them, determinism allows to set connection between the initial conditions of the system and state of this system in a time of T (even if T aspires to endlessness). State of the physical system precision determination in the arbitrary moment of time is simply related to exactness of task of initial conditions. Consequently, in general case physical objects are considered different, if difference in the values of physical sizes it is possible to discover in $n+1$ a sign after a comma, even if $n \infty 0$. In this case, exactness of determination of value of physical size depends only on exactness of task of initial conditions of measuring experiment. Conducting successive clarifications of these initial conditions and, the same, carrying out the increase of measuring exactness, it is possible to attain the absolute value of physical size. As within the framework of classic determinism, determination of physical size is foreseen only through maximum transition, on this account the concept of truth value which is the result of maximum transition was entered in the theories of measuring. As the truth value is a base concept at construction of theory of error of measuring, it is possible to assert that classic determinism is the theoretical base of error of measuring.

2.2. Physically justified determinisms in the theory of measuring

Studying the problems of the physical raising of tasks within the framework of classic determinism Max Born entered the concept of the physically justified determinism, and showed that the decision of the simplest determined task must be carried out subject to the condition initial, set in a probabilistic form. The just the same raising of task corresponds to the real physical situations.

Leaning against development of classic determinism in the direction of application of probabilistic description of values of physical sizes for the real estimation of the state of the physical systems, it is possible to build the theory of analysis of results of measuring, different from the existent theory of error of measuring.

As already on the stage of description of the real physical situation of measuring process application of principles of determinism came across impossibility of their realization in full, the simplified model of determinism was used in practice of measuring. However high-quality this model not money-changer of approach to the process of measuring and analysis of their results.

The theory of error developed in the total, but of principles base concepts had not changed. Therefore possibility to revise the physical grounds and theories of error of measuring appeared with appearance of concept of vagueness of results of measuring.

Application of physically intelligent determinism in the theory of measuring is related to that any bodily condition can be known with small, but always by eventual inaccuracy. Therefore, basic data, both for the direct measuring and for indirect, must be set by a not number, and probabilistic distributing.

Development of concept of vagueness in measuring is linked with the use of probabilistic all parameters specification in equalization of measuring that is mathematically described by a pdf function.

Thus, the vagueness of result of measuring exists always, even in default of casual influence of environment on a measuring process, as there is a vagueness in the task of initial conditions for the determined equalization of measuring. And in this sense the vagueness of result of measuring can be examined as development of theory of error of measuring and which allows more fully to take into account the real physical situations of measuring procedures.

How additional confirmation of necessity of the use of concept of vagueness at the analysis of results of measuring it is possible to examine possibility of existence in the determined dynamic systems of irregular, stochastic motion. Such conduct results in impossibility exactly to estimate the parameters of the system, as such values are not present.

2.3. Chance and stochastic in the determined systems

Analysing the terms of origin of vagueness of result of measuring, a conclusion was done, that initial parameters in equalization of measuring it is necessary to represent in a probabilistic kind, the mechanism of casual conduct of physical sizes is not examined here, assuming that he can be linked either with influencing of environment or the values of these parameters are certain, in same queue, with some vagueness.

Whatever the source of vagueness of parameters of the explored system was, the task of study of this system is examined as determined with casual initial conditions. The determined decision based on casual initial conditions has the variation conditioned by character of decision of this determined equalization subject to the condition initial different.

On principle new in conducting of measuring in the dynamic systems there is existence in the nonlinear dynamic systems of autoswaying character of the mode of "dynamic chaos". We will mark that dynamic chaos is one of forms of development of synergetic effects. Presence of such modes to stipulate casual change of values of physical sizes in such systems, and, as a result, necessity to take into account during conducting of measuring extremity of interval of their casual variations.

Not jutting out into the detailed analysis of reasons of origin of such modes we will mark that subject to the condition initial fixed the decision of the determined task has casual variation exceeding exactness of task of initial conditions. Consequently the pattern of behavior of the explored system brings in a vagueness in estimation of its parameters.

As development of dynamic chaos forms the function of distributing closeness, incident to this system, the analysis of results of measuring must be carried out after verification of the realized casual process on ergodic.

Application of probabilistic methods for the analysis of physical sizes and results of their measuring possibly on condition, that the studied statistical processes are ergodical.

When we investigate dynamic systems with periodical behavior and must measure parameters of these systems, we have to take into account the character of dynamical changes. Now, it is well known, that nonlinear dynamical systems (physical or chemical) may have deterministic and chaotic behaviors. An example of the chaotic behavior is strange attractor [2]**.** In case of deterministic behavior we have a type of well-investigated dynamic measurements, but we have not methods for estimated the results of measurement in case of dynamic chaos [3, 4].

Main aim of this work directs on the investigation of measurement uncertainty transformation under condition that changes of measurand value connected with chaotic dynamic behavior in system.

For the realization of this aim in paper was using mathematical model based on nonlinear discrete - difference equation that are well-known in literature as logistic equation [5]. This equation has some different types of solutions. One type of this solutions, describes deterministic periodical behavior in time, while other type of solutions describes chaotic or non-regular behavior. It was necessary to make clear how statistical properties of chaotically solutions influence on value uncertainty.

So, the purpose of the present work was investigation influence different solutions of mathematical model, based on discrete difference equation, on measurement uncertainty [5].

3. METHOD

GUM establishes that the measurand X is defined if the functional relationship or the measurement model is given

$$X = F(\vec{\mu}) \ , \qquad (1)$$

where $\vec{\mu} = (\mu_1, \ldots \mu_N)$ is a N -dimension vector of input quantities.

In dynamic systems, the measurement model is given by a system of differential equations [6]

$$\frac{\partial \vec{x}}{\partial t} = f(\vec{x}, \vec{\mu}) \ , \qquad (2)$$

where $\vec{x}$ is a vector of measurands, $\vec{\mu}$ is a vector of input quantities and $f(\vec{x}, \vec{\mu})$ - nonlinear function.

Using (2) as the measurement model, we can determine conditions, under which the dynamic system's behavior will influence on the measurement results and measurement uncertainty.

In accordance to GUM, the measurement of quantity x as a single (scalar) output quantity has physical sense in two cases.

In the first case the system under investigation is in stable and stationary state, i.e.

$$\frac{\partial x}{\partial t} = 0 . \quad (3)$$

In this situation GUM measurement model based on a nonlinear algebraic equation

$$f(x, \mu) = 0 . \quad (4)$$

Often, in measurement practice, a simpler equation than equation (1) is used [1]

$$x - f(\mu) = 0 . \quad (5)$$

Usually, the system of differential equations (2) was using for describing dynamic system's behavior. In our second case, the dynamic system, are in state with deterministic behavior in time. Dynamic systems with different behavior deterministic or chaotic, described by discrete difference equation is used [5]

$$x_{n+1} = G[x_n, \mu], \quad (6)$$

where G is a nonlinear function.

The non-linearity and non-equilibrium are two properties that fully determine deterministic chaotic behavior of the dynamic system. While statistical properties of deterministic chaos are well known, there is no proof that all types of deterministic chaos are ergodic random processes. So, the application of GUM's recommendations for the estimation of measurands and its uncertainty requires investigations of the properties of deterministic chaos and its influence on the measurement of uncertainty in every separate case.

In special case of square one-dimensional mapping (6) that is known as the logistic mapping equation

$$x_{n+1} = \mu x_n (1 - x_n), \quad (7)$$

solutions of (7), corresponding to actual states of a physical system, are attracting and limiting points or cluster sets. Only the limit of a sequence x_n presents the practical interest for measurement

$$x(\mu) = \lim_{n \to \infty} x_n(\mu) . \quad (8)$$

Accordingly, in the result of measuring we determine the numerical value of $x(\mu)$, which depends on parameter μ. So, all cases of the measurement models are based on (8) and we have to measure μ for definition of $x(\mu)$. In this case we can use GUM recommendations.

Depending on a numerical parameter value μ (in an interval from 0 up to 4) the limits in (8) can have different physical sense. And it, in turn, can result in to different results at the measuring quantity $x(\mu)$. The basic problem, which in operation was necessary for clarifying in connection with sensitivity of values $x(\mu)$ from quantity μ, encompassed in definition of requirements, at which the system behavior renders influence to statistical scattering measurement results.

The solutions of the equation (8) are well investigated (see for example [3]), therefore, in the present operation these solutions were considered as the equation of measuring of quantity $x(\mu)$ at different values μ. Basing on these equations, the estimation of probable statistical scattering measurement results were carried out depending on values μ.

The solutions (8) of the equation (7) are well investigated (see, for example [5]), therefore, in the present consideration these solutions were treated as the measuring equation for $x(\mu)$ at different values of μ. Basing on these equations, which have different physical senses, the estimation of probable statistically- spreaded measurement results were carried out for different values of μ.

Because of the sensitivity of values $x(\mu)$ to values of μ, the basic measurement problem, which is necessary to solve, is defining on the conditions, at which the system behavior influence the statistical spreading of measurement results.

By using the logistic mapping equation (7), as measurement model, let us consider some well-known solutions for different physical situations for values of μ in interval from 0 up to 4.

1) $0 < \mu \le 1$. The solution (7) represents an inconvertible attractive fixed point $x = 0$. In this case the measuring of value of quantity x presents a trivial problem.

2) $1 < \mu \le 3$. The square-law mapping has a unique fixed limiting attractive point $x^{(1)}$, which value is determined by the expression

$$x^{(1)} = 1 - 1/\mu , \quad (9)$$

for any initial value x_0.

This solution of (7) is the limiting value of the mapping and presents the measurement equation for quantity $x^{(1)}$. Therefore, utilizing results of direct or indirect n measurements of the parameter $\mu_k, k = 1,...n$, the value of $x^{(1)}$ is determined that is relevant to a stable and attractive conditions of system. The mean value $\overline{x^{(1)}}$ and the standard deviation $u_c(x^{(1)})$ can be expressed as follows:

$$\overline{x^{(1)}} = \frac{1}{n} \sum_{k=1}^{n} (1 - 1/\mu_k), \quad (10)$$

$$u_c(x^{(1)}) = 1/\mu^2 u_c(\mu), \quad (11)$$

where

$$u_c(\mu) = \sqrt{\frac{1}{n(n-1)} \sum_{k=1}^{n} (\mu_k - \overline{\mu})^2} . \quad (12)$$

Thus, the measurement of uncertainty of $x^{(1)}$ depends only on the measurement uncertainty of the parameter μ.

3) $3 < \mu \le 3.44948...$ The solution of the equation (8) has two inconvertible fixed points, which values are determined by a value of the parameter μ.

$$x^{(2),(3)} = \left[\mu + 1 \pm \left(\mu^2 - 2\mu - 3\right)^{1/2}\right] / 2\mu \ . \qquad (13)$$

As well as in the previous case the medial results of measuring the quantities of inconvertible points $x^{(2,3)}$ and estimation of measurement uncertainty of the results in these measuring are determined by the results of measurement of the parameter μ .

4) 3.5699 < μ ≤ 4. The transformation (7), implemented at indicated values of the parameter μ , has a limit representing a cluster set, which sizes are determined by the field of existence of the solutions of this equation. The structure of this set is determined by a parameter value μ . In [2] is shown, that inside a single quadrate of a define area of quantity x_n , there is a random change of values of quantities x_n . It signifies, that everyone next itteration gives in value of quantity x_n , which by a casual fashion differs from previous x_{n-1} . It signifies, that at any starting conditions x_0 result sequentially of the fulfilled iterations in (8), when $n \to \infty$, is the sequence of random values x_n . On the other hand, the existence of such solution (8), characterizes absence of the determined equation of measuring in that view, in what they represented at other values μ .

Thus, the result of viewed square-law mapping becomes not an inconvertible limit point characterized by a unique value, but range of values x_n , in which the random values x_n are featured by a probabilities frequency function - $\varphi(x_n, \mu)$. The properties of square-law mapping are those, that at any value μ_i the probability density function $\varphi(x_n, \mu_i)$ is shaped. At enough great many of iterations the allocation of values of quantity x_n becomes practically continuous, that allows to proceed to continuous functions density of probability - $\varphi_i(x, \mu_i)$.

Under this values of the parameter μ , the limit (8) is cluster set, which sizes are determined by the region of existence of the solutions of this equation. The structure of the set is determined by a value of the parameter μ . In [5] it is shown, that inside a unit square of a region of definition of quantity x_n , there is a random change of values of quantity x_n . It means, that every next iteration gives value of quantity x_n , which randomly differs from previous value of quantity x_{n-1} . It means, that at any initial condition x_0 is the result of the sequentially fulfilled iterations of equation (7), when $n \to \infty$, is the sequence of random values x_n . On the other hand, the existence of such solution of equation (7) indicates the absence of the determinate measuring equation in the sense, in which the equation (7) did serve at other values of μ .

Thus, the solution of considered square-law mapping becomes not an inconvertible limit point characterized by a unique value, but range of values x_n , in which the random values x_n is defined by a probability frequency function - $\varphi(x_n, \mu)$. The properties of the square-law mapping are such that at any value of μ_i the probability density function $\varphi(x_n, \mu_i)$ are shaped. At great many enough of iterations the distribution of values of quantity x_n becomes practically continuous that allows proceed to continue probability density function - $\varphi_i(x, \mu_i)$.

At each value of μ_i the mean value $\overline{x_i}$ will be determined by the relevant probability density function

$$\overline{x_i} = \int_0^1 x\varphi_i(x, \mu_i)dx \ . \qquad (14)$$

The feature of this equation is that the value $\overline{x_i}$ depends on sort of a function φ_i which, in turn, exhibits strong sensitivity to value of the parameter μ_i . That is, the small change in value of this parameter gives an essential change to function $\varphi_i(x, \mu_i)$ character, that in turn, gives an essential change in value of $\overline{x_i}$. It means, that the value of difference $\overline{x_i} - \overline{x_j}$ at close values of μ_i and μ_j can differ by an arbitrary random amount. It means, that at the normal law of spreading of the apparent parameter's values of μ , the variance of values $\overline{x_i}$ will be anomalously large.

If m of the observations of the quantity μ_i are carried out, the mean value of the result of measuring of quantity $\overline{x}$ is determined by the expression

$$\overline{x} = \frac{1}{m}\sum_{i=1}^{m}\overline{x_i} = \frac{1}{m}\sum_{i=1}^{m}\int_0^1 x\varphi_i(x, \mu_i)dx \ . \qquad (15)$$

In the result of measuring stipulated by the spreading of measured values of μ , the variance is determined by the expression

$$u_c^{\,2} = \frac{1}{n(n-1)}\sum_{i=1}^{m}(\overline{x_i} - \overline{x})^2 \ . \qquad (16)$$

Thus, the result of observation of the statistical spreading of values of driving parameter μ is magnified by therandom dynamics of the square-law mapping and this gives abnormal magnification of the uncertainty in the estimation of $\overline{x}$.

With the understanding that gives the acknowledgement of the random behavior of the dynamic system at $n \to \infty$, we shall consider more in details sampled iterations of the mapping (7). Thus, the possible conditions of the practical implementation of the process of measuring of quantity x_n depend on an interval τ between observations of the quantities x_n and number of observations N . If T is an interval between sequential iterations of the equation (7), it is feasible to consider two extreme cases: first is for $N\tau \leq T$, and second is for $N\tau > T$.

If the condition $N\tau \leq T$ is satisfied, the mean value $\overline{x_n}$ with standard deviation $u_c(x_n)$ will be set on n -th iteration of square-law mapping which is stipulated with the uncertainty $u_c(\mu)$. At the next iteration of mapping the other value of x_{n+1} will be measured, which also will be carried out with the uncertainty $u_c(\mu)$. And each time, after the next iteration of the mapping, it will be measured randomly modified value of x_n . Nobody can ensure the reproducibility of measuring in the square-law mapping operating in a random mode since the physically stable values of observed data do not exist. Therefore, a measure of uncertainty is necessary for observed data generated randomly within a region of finite values.

If the interval τ between measuring of the quantity x_n is more than a time slice T , the series of random values x_n will be captured in result of N observations. Thus, as it was shown above, the uncertainty of the result, both single observation and result of measuring, will be stipulated by the stochastic dynamics of the behavior of the measurand. As the region of random change of quantity x_n is restricted, it is possible to view this region as limiting, stable borders of explored system, which "is spread" statistically in finite region.

4. RESULTS

Basing on these equations, which have different physical senses, the estimation of probable statistically- spreaded measurement results were carried out for different values of μ . Because of the sensitivity of values $x(\mu)$ to values μ , the basic measurement problem, which is necessary to solve, is defining the conditions, at which the system behavior influence the statistical spreading of measurement results.

Under the values of the μ parameter $3.5699 < \mu \leq 4$, the limit (8) is cluster set, which sizes are determined by the region of existence of the solutions of this equation. The structure of the set is determined by a value of the μ parameter. In [5] it is shown, that inside a unit square of a region of definition of quantity x_n , there is a random change of values of quantity x_n .

For an example we will consider a random sequence values x_n , generated by the square-law mapping (3) at μ =4. In [5] the statistical property of the sequence is characterized by a probability density distribution function $P_4(x)$ (coefficient 4 points at parameter value μ =4)

$$P_4(x) = \frac{1}{\pi}\frac{1}{\sqrt{x(1-x)}} . \quad (17)$$

This function is defined in an interval of values $0 < x < 1$. It was taken into account that the stochastic process implemented at μ =4 with intermixing and exponential divergence of close trajectories [5], is ergodic.

Utilizing function (15), it is possible to calculate the mean value of the dynamic quantity $\overline{x_4}$

$$\overline{x_4} = \frac{1}{\pi}\int_0^1 \sqrt{\frac{x}{1-x}}dx = \frac{1}{2} , \quad (18)$$

and the standard deviation

$$S = \sqrt{\frac{1}{\pi}\int_0^1 \frac{(x-1/2)^2}{\sqrt{x(1-x)}}dx} = 0{,}353. \quad (19)$$

Thus, even at usage of an absolutely precise parameter value μ (in a surveyed case μ =4), the one-dimensional mapping shapes a casual sequence, x_n , which is characterized by a probability density function of allocation (15), with this help it is possible to calculate average value and reference diversion.

From here follows, that at development of a random mode in behaviour of dynamic system there is a key feature, which consists that stationary unique value for a measurand primely does not exist. Therefore indeterminacy of the result of measuring of the parameters of the system which is taking place in a random mode, will be stipulated by a casual straggling of the values of a measurand.

5. DISCUSSION AND CONCLUSIONS

Utilizing one-dimensional discrete mapping for exposition of the character of the behavior of dynamic system, it is shown how the stochastic behavior of the explored dynamic system can influence an estimation of system parameters from the observed data. It is shown that the statistical character of the behavior of the dynamic system is the decisive factor for estimation of the parameters of the system from the observed data. On a theoretical example of one-dimensional logistic mapping it is shown the necessity of the acknowledgement of the random behavior of the dynamic system for the estimation of the observed data. Viewing introduced in the present operation of the examination as one of the perspective directions of the development of the theory of measurings, including methods of an error estimation or indeterminacy of the fulfilled measuring, is possible to formulate a blanket problem, which solution is represented to extremely necessary now.

Applied examinations of the composite dynamic systems, by which it is possible to refer of a different sort biological objects [5], social systems, objects with a composite interior kinetics of chemical reactions, and also all systems based on the thermodynamic nonequilibrium processes [6], should be grounded on the effects basic researches of the synergeticcs effects (to them the processes of the self-organizing resulting in to the development of the spatial, time and time-multiplexed structures and processes of the dynamic chaos causing irregular, casual behavior of the dynamic systems), explicating in these systems concern.

In this time the explicated methods of the experimental examinations of dynamic chaos are not methods of an estimation of the observed data, which ensure unity of measuring. And for this reason, the examinations of the influence of the dynamic chaos on the observed data in the dynamic systems should have a metrological directedness, which can provide the development of the uniform approaches for measuring problems.

Examinations applied to complex dynamic systems (such as different sort biological objects, social systems, objects with a composite interior kinetics of chemical reactions, and also all systems based on the thermodynamic non-equilibrium processes should take into account the fundamental synergetic effects (such as self-organizing resulting in the development of the spatial, time and time-multiplexed structures and processes of the dynamic chaos causing irregular, random behavior of the dynamic systems, explicating the behavior of these systems.

ACKNOWLEDGMENTS

This report had financial support by INTAS project Ref Nr 06-1000024-9075 Project Title “Development of new atomic clocks using optical frequency comb”

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