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ALTERNATIVE METHODS TO ESTIMATE MEASURAND VALUES: MODELS AND OPERATIVE IMPLICATIONS

Luca Mari [1], *David Macii* [2]
[1] Università Cattaneo – LIUC, Castellanza (Va), Italy, lmari@liuc.it
[2] University of Trento – DISI, Trento, Italy, macii@disi.unitn.it

Abstract: This paper compares two alternative averaging methods commonly adopted to estimate measurand values, in agreement with the guidelines of the *Guide to the Expression of Uncertainty in Measurement* (GUM). According to the proposed analysis, validated in a simple case study, the choice of best estimation method should depend not only on the amount of nonlinearity of the measurement model, but also on the amount of definitional and acquisition uncertainty of the input measurands.

Keywords: measurand value, measurement uncertainty, foundations of measurement.

1. INTRODUCTION

The *Guide to the Expression of Uncertainty in Measurement* (GUM) [1] introduces a seldom emphasized constraint on the measurand. In fact, the par. 1.2 of the GUM states that "this *Guide* is primarily concerned with the expression of uncertainty in the measurement of a well-defined physical quantity – the measurand – that can be characterized by an essentially unique value." This is plausibly the main reason why measurement results are formally expressed by two terms, namely:

- a single value representing the measurand;
- the corresponding standard uncertainty, estimated as the standard deviation of the average of the sampling probability distribution of the quantity to be measured.

From this formalization, the GUM also derives the concept of expanded uncertainty and the related interval-based notation.

The attention of most researchers studying fundamental measurement principles has been recently focused on standard uncertainty estimation techniques, under the implicit assumption that the characterization of the measurand estimator does not require any further discussion. On the contrary, we claim that the analysis of best estimator for the measurand value arises fundamental philosophical and operative issues. In particular, this paper deals with the conceptual and formal relations between the (non-)uniqueness of the measurand value and the (non-)linearity of the so-called measurement model function, and shows that sometimes the assumption of the GUM about the uniqueness of the measurand value is not appropriate.

2. MODELS IN MEASUREMENT: ON THE UNIQUENESS OF MEASURAND VALUE

An important contribution of the GUM is its reconsideration of the classical distinction between "direct" and "derived" (or "indirect") measurements. Given that several components, from both Type A and Type B evaluations, generally contribute to the measurand uncertainty, *any* measurement in which such components must be combined is indeed an indirect operation. While we fully agree with this standpoint, denoting the (analytical) expression $f(.)$ that relates the "input measurand(s)" $\boldsymbol{X}$ to the "output measurand" Y as *the* (mathematical) measurement model is somehow questionable.

In fact, two models should be considered in the interpretation of measurement process and measurement results:

- the *measurand model*, based on some "background knowledge" about the measurand itself (e.g., obtained from Physics);
- the *measurement model*, describing the measurement process as constituted of both an acquisition stage, aimed at obtaining a value for the input measurands, and a processing stage, returning a value for the output measurand.

The acknowledgment that these models unavoidably provide only approximate interpretations of the measurands and the measurement process respectively (a model-based version of the classical unknowability of "true values") is conceptualized and formalized in terms of *uncertainties*. In fact, the *intrinsic uncertainty* (or *definitional uncertainty*, as now it is called in [2]) for each input measurand is part of the measurand model, whereas the corresponding *acquisition uncertainty* derives from the measurement model. Notice that even though this second model might also include some uncertainty contributions due to the processing stage, we will not consider this issue in the following (as also the GUM does), because it is not significant for our purposes. Consider also that the definitional uncertainty results "from the finite amount of detail in the definition of a measurand" [2], as possibly due to some additional contribution superimposed to the quantity to be measured, and is not to be confused with the possible time-varying behavior of the measurand itself.

A synthesis of this interpretation about the models mentioned above and their relationships with the possible uncertainties is shown in Figure 1.

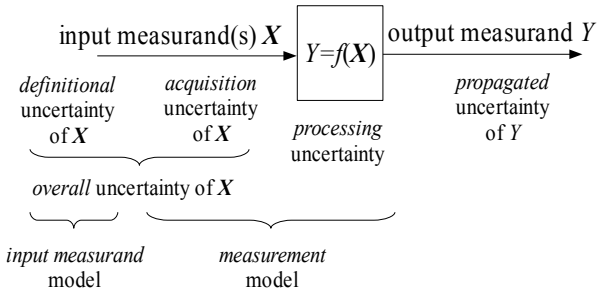

Fig.1 – Models involved in measurement and their relation to uncertainties.

In fact, the assumption related to the "essentially unique value" of the measurand stated in the paragraphs 1.2 and 3.1.3 of the GUM [1], as well as the discussion about its possible coincidence with the "true value", is clearly related only to the measurand model, and therefore affects only the definitional uncertainty. This interpretation is confirmed by the new edition of the VIM [2], that in the definition of "true quantity value" (2.11, Note 3) points out that "when the definitional uncertainty associated with the measurand is considered to be negligible compared to the other components of the measurement uncertainty, the measurand may be considered to have an 'essentially unique' true quantity value. This is the approach taken by the GUM and associated documents, where the word 'true' is considered to be redundant.".

Furthermore, such a uniqueness requirement does *not* imply that the measurand has to be characterized by a single real number (sometimes referred to also as *singleton*). As a cogent counterexample, if the range of the "model function" $f(.)$ is the power set of the real set $\boldsymbol{R}$, any measurement result is basically a *unique* interval of *infinite* real numbers. Hence, "unique value" does not formally mean a single real value: the condition that the measurand value is a real number does not follow from the assumption of the GUM, although it is very plausible that such a condition is the actual requirement meant by the GUM. There are, of course, at least two good reasons justifying this condition [3, 4], i.e.:

- one based on ontology: the Pythagorean-Platonic-Galileian-Laplacian-... viewpoint assumes that "numbers are in the world", and that those numbers are rationals (and reals, since the XIX century);
- one based on pragmatics: the equations formalizing physical laws are formulated to be naturally and efficiently applied to real numbers.

How should the condition of uniqueness be interpreted in the realistic situation in which the definitional uncertainty is not null?

Whereas the GUM suggests assuming the definitional uncertainty as a component of the uncertainty budget (e.g., by simply listing the "incomplete definition of the measurand" among the "possible sources of uncertainty in a measurement", as stated in the par. 3.3.2), we claim, in accordance with what is stated in [5], that the comparison between definitional uncertainty and measurement uncertainty (in the following interpreted as acquisition uncertainty) operates as a discrimination factor:

- if the definitional uncertainty is negligible with respect to the acquisition uncertainty, the measurand itself can be regarded as a real-valued continuous variable. We will refer to this case as *Case A*;
- if, on the other hand, the definitional uncertainty is larger than the acquisition uncertainty, then the uniqueness condition becomes critical, particularly when the model function $f(.)$ is not linear. We will refer to this case as *Case B*.

It should be noted that, as a consequence of this model-dependent interpretation, the choice of assuming a unique value (actually: a real value, as commented above) is only pragmatic, with nothing "essential" in it, as instead claimed by the GUM.

By means of a simple example, in the following we will analyze these two cases separately, and will propose some suggestions about how to manage the effects of both negligible and large definitional uncertainties.

3. DEFINITIONAL AND ACQUISITION UNCERTAINTY IN MEASURAND ESTIMATION: A CASE STUDY

A data acquisition board is used to collect multiple instantaneous samples of a voltage signal applied to a reference resistor having a nominal resistance of 1 Ω with negligible definitional uncertainty. If the quantity to be measured is the power p dissipated by the resistor and the nominal applied voltage v is constant over time, two distinct situations may occur:

- if the applied voltage has negligible definitional uncertainty with respect to the acquisition uncertainty (*Case A*), the result of the data acquisition stage can be described by the random variable $V_A=v+N_{acq}$, where N_{acq} is the random variable modeling the uncertainty of the measurement process;
- if the applied voltage results from the superposition of a constant value v and some significant definitional uncertainty N_{def} (e.g., due to a 50 Hz sine wave interference) (*Case B*), the result of the data acquisition stage can be described by the random variable $V_B=v+N_{def}+N_{acq.}$ Note that the definitional uncertainty described in this example has nothing to do with the fact that the underlying phenomenon is possibly time-variant; in fact, no transient effects are taken into account here, and the system is assumed to be observed in stationary conditions.

In the following, both N_{acq} and N_{def} are assumed to be uncorrelated white noises, with zero mean and variance σ^2_{acq} and σ^2_{def}, respectively.

A major difference exists between Case A and Case B. In the first one, since the measurement uncertainty is negligible the measurand value p results from:

$$p=E[v^2]=v^2 \tag{1}$$

where $E[\cdot]$ is the operator returning the expected value of the argument. On the contrary, in the second case the measurand value is given by:

$$p=E[(v+N_{def})^2]=v^2+\sigma^2_{def} \tag{2}$$

Such a difference between Cases A and B may have serious consequences when the two averaging methods suggested in the GUM to reduce the measurement uncertainty are applied to measurement results. Assume, for instance, that K samples V_{X_k}, k=1,…,K, of the quantity V_X (where the

subscript X is either A or B depending on the specific case considered) are collected by the data acquisition system. According to [6], the measurand value can be estimated using one of the two following alternative methods, i.e.:

- *Method 1*: by averaging the acquired samples before applying the square function:

$$\hat{P}_{X,1}=\left(\frac{1}{K}\sum_{k=1}^{K} V_{X_k}\right)^2 \tag{3}$$

- *Method 2*: by averaging the instantaneous power values:

$$\hat{P}_{X,2}=\frac{1}{K}\sum_{k=1}^{K} V_{X_k}^2 \tag{4}$$

According to the adopted notation, $\hat{P}_{A,1}$, $\hat{P}_{A,2}$, $\hat{P}_{B,1}$, and $\hat{P}_{B,2}$ are the random variables modeling the measurand estimators obtained in Cases A or B and by applying Methods 1 or 2, respectively. The following table summarizes the meaning of the Cases and of the Methods considered.

Table 1 – Cases and Methods involved in the case study.

Case	A	negligible definitional uncertainty
	B	considerable definitional uncertainty
Method	1	samples are averaged and then $f(.)$ is applied to the result
	2	$f(.)$ is applied to each collected sample and then the results are averaged

In the Appendix A it is shown that if the K collected samples are statistically independent, the mean values and the variance of the estimators $\hat{P}_{A,1}$, $\hat{P}_{A,2}$, $\hat{P}_{B,1}$, and $\hat{P}_{B,2}$ are given respectively by:

$$E[\hat{P}_{A,1}]=v^2+\frac{\sigma_{acq}^2}{K} \tag{5}$$

$$E[\hat{P}_{A,2}]=v^2+\sigma_{acq}^2 \tag{6}$$

$$E[\hat{P}_{B,1}]=v^2+\frac{\sigma_{def}^2+\sigma_{acq}^2}{K} \tag{7}$$

$$E[\hat{V}_{B,2}]=v^2+\sigma_{def}^2+\sigma_{acq}^2 \tag{8}$$

and

$$\boldsymbol{var}[\hat{P}_{A,1}]=4v^2\frac{\sigma_{acq}^2}{K}+\frac{\gamma_{acq}}{K^3}+\left(\frac{2K-3}{K^3}\right)\sigma_{acq}^4 \tag{9}$$

$$\boldsymbol{var}[\hat{P}_{A,2}]=4v^2\frac{\sigma_{acq}^2}{K}+\frac{\gamma_{acq}-\sigma_{acq}^4}{K} \tag{10}$$

$$\boldsymbol{var}[\hat{P}_{B,1}]=4v^2\frac{\sigma_{def}^2+\sigma_{acq}^2}{K}+\frac{\gamma_{def}+\gamma_{acq}+6\sigma_{acq}^2\sigma_{def}^2}{K^3}+$$
$$+\left(\frac{2K-3}{K^3}\right)\left(\sigma_{def}^4+\sigma_{acq}^4+2\sigma_{def}^2\sigma_{acq}^2\right) \tag{11}$$

$$\boldsymbol{var}[\hat{P}_{B,2}]=4v^2\frac{(\sigma_{def}^2+\sigma_{acq}^2)}{K}+$$
$$+\frac{\gamma_{def}-\sigma_{def}^4+\gamma_{acq}-\sigma_{acq}^4+4\sigma_{acq}^2\sigma_{def}^2}{K} \tag{12}$$

where $\boldsymbol{var}[.]$ is the variance operator and γ_{def} and γ_{acq} are the 4th-order moments of the definitional and acquisition noise components, respectively. Moreover, if $v^2 \gg \sigma_{acq}^2$ and $v^2 \gg \sigma_{def}^2$, as it usually occurs in practice, equations (9), (10), (11), and (12) can be approximated as:

$$\boldsymbol{var}[\hat{P}_{A,1}]\simeq\boldsymbol{var}[\hat{P}_{A,2}]\simeq 4v^2\frac{\sigma_{acq}^2}{K} \tag{13}$$

and:

$$\boldsymbol{var}[\hat{P}_{B,1}]\simeq\boldsymbol{var}[\hat{P}_{B,2}]\simeq 4v^2\frac{\sigma_{acq}^2+\sigma_{def}^2}{K} \tag{14}$$

4. GENERAL CONSIDERATIONS

In general, if $f(\cdot)$ weakly nonlinear, the variance of both estimators (3) and (4) is approximately the same. Nevertheless, such estimators are not always unbiased. In fact, by comparing the equations (5)-(8), it can be clearly observed that in Case A the result of Method 1 converges asymptotically to the quantity to be measured (i.e., $p=v^2$) as $K\to\infty$, whereas in this case the estimate obtained using Method 2 is biased by the noise power associated to the acquisition process (a similar conclusion is also reached in [6]). On the contrary, in Case B both estimators are biased. However, while the asymptotic bias $-\sigma_{def}^2$ introduced by Method 1 cannot be controlled or reduced by the user, the acquisition noise power biasing the result of Method 2 can be made negligible by using a data acquisition system such that $\sigma_{acq} \ll \sigma_{def}$.

In conclusion:

- Method 1 is asymptotically unbiased if the definitional uncertainty is negligible with respect to the acquisition uncertainty;
- Method 2 is instead preferable whenever the uncertainty affecting the measurement process can be made negligible compared to definitional uncertainty.

The distinction between Methods 1 and 2 is implicitly acknowledged by the GUM itself. In fact, in the note to par. 4.1.4, the GUM states that, while the estimate of the output quantity y should result from:

$$Y_1=f\left(\frac{1}{K}\sum_{k=1}^{K} X_k\right) \tag{15}$$

in some cases the estimate of y may be preferably obtained from:

$$Y_2=\frac{1}{K}\sum_{k=1}^{K} f(X_k) \tag{16}$$

Of course, the GUM is correct in stating that "the two approaches [i.e., Methods 1 and 2] are identical if $f(\cdot)$ is a linear function of the X_k". The estimation of the measurand value performed according to (15) (i.e. Method 1) is generally computationally more efficient than the one

resulting from (16) (i.e., Method 2). In fact, in the former case the function $f(\cdot)$ is applied just once, whereas in the latter it must be computed K times. This justifies the operative preference given to Method 1, whenever linearity can be assumed.

On the other hand, in reference to Method 2 the GUM states that "this way of averaging may be preferable when $f(\cdot)$ is a nonlinear function of the input quantities $X_1,...,X_N$" ([1], 4.1.1, Note 1), while not making the conditions of choice between the two averaging ways explicit.

5. SIMULATION RESULTS AND DATA ANALYSIS

The previous theoretical results have been validated by means of some Monte Carlo simulations, by applying both Methods 1 and 2 to Cases A and B. The details of the algorithm used to run such simulations are shortly reported in the Appendix B. In all simulations, a voltage signal of nominal value 5 V with a superposed white, zero mean, Gaussian acquisition noise has been applied to a reference resistor with a nominal resistance of 1 Ω. Each scenario has been simulated by performing 2000 runs for each number of samples K, ranging from 10 to 1000 with a step of 20. For each value of K, $\hat{P}_{X,1}$, $\hat{P}_{X,2}$ have been estimated and their means and variances have been computed. Such statistics have been also compared with the values given by equations (5)-(8) and (13), (14), in order to check the correctness of the theoretical analysis.

The mean and variance values obtained in Case A (i.e., when the definitional uncertainty is negligible) and with σ_{acq} =0.5 V are shown in Figures 2 and 3, respectively.

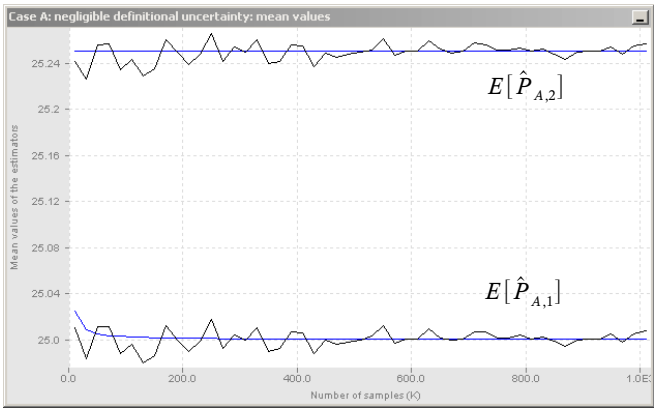

Fig.2 – Mean values of the estimates involved in Case A.

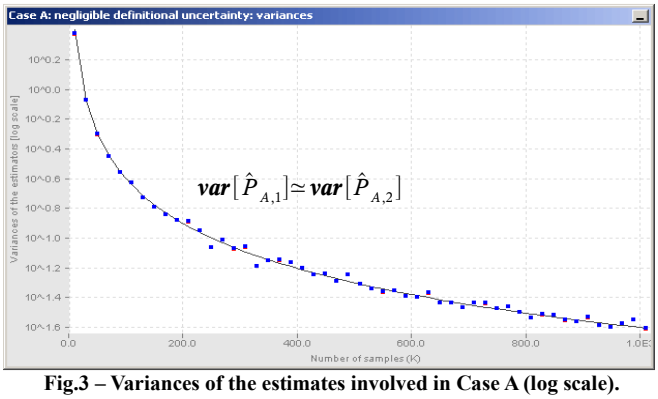

Fig.3 – Variances of the estimates involved in Case A (log scale).

The line resulting from (5) and the corresponding Monte Carlo simulation pattern based on Method 1 are shown in the lower part of Figure 2. Notice that both curves converge to the nominal value given by (1) (horizontal dotted line) when the number of samples $K \to \infty$. On the contrary, the results of the Monte Carlo simulations performed according to Method 2 and the theoretical values obtained from (6) are shown in the upper part of the same figure. Observe that such results are biased by a factor σ^2_{acq}, as expected.

The chart in Figure 3 shows the line obtained from equation (13), together with the results of the Monte Carlo simulation performed according to Methods 1 and 2, displayed as dots. Notice that the dotted lines related to the simulation results are almost indistinguishable from the theoretical curve, thus confirming the assumptions underlying (13).

Figures 4 and 5 are similar to the previous ones and they show the mean values and variances with respect to the corresponding theoretical values in Case B, by assuming that σ_{acq} =0.1 V and σ_{def} =0.5 V, i.e., larger than the acquisition uncertainty.

The line resulting from (7) and the corresponding simulation pattern based on Method 1 are shown in the lower part of Figure 4, whereas the line resulting from (8) and the simulation pattern based on Method 2 are in the upper part of the same figure. In this case the actual measurand value is defined by equation (2) and is represented as an horizontal dotted line. Thus, Method 1 introduces a $-\sigma^2_{def}$ bias for $K \to \infty$, while the result of Method 2 is affected by a constant σ^2_{acq} bias that can be possibly minimized by suitably reducing the acquisition uncertainty.

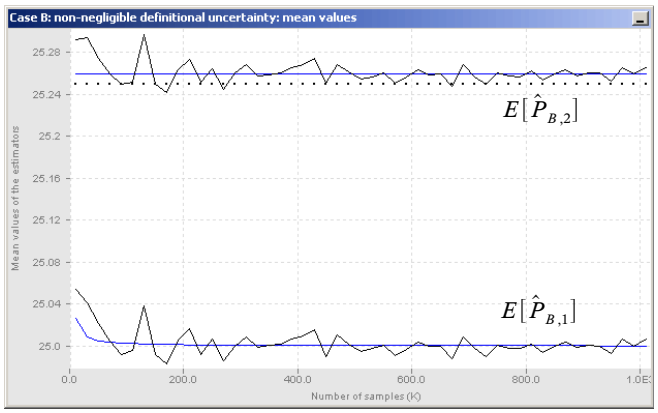

Fig.4 – Mean values of the estimates involved in Case B.

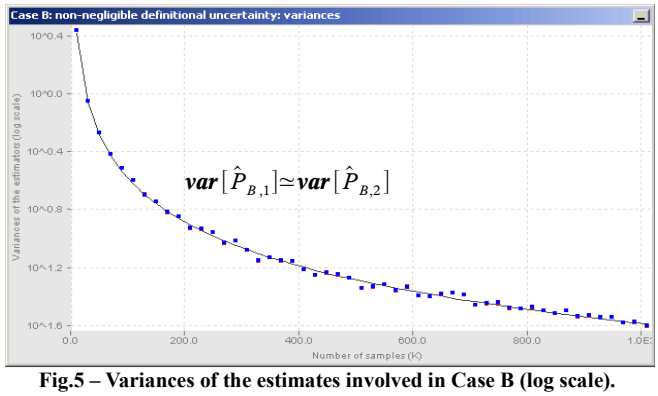

Fig.5 – Variances of the estimates involved in Case B (log scale).

Finally, the chart in Figure 5 shows the line obtained from equation (14), together with the results of the Monte Carlo simulations performed according to Methods 1 and 2. As in Case A, the simulation results are in excellent agreement with the theoretical assumptions.

6. CONCLUSIONS

In this paper, we proved that the preferable averaging method used to estimate the measurand in case of repeated measurements should rely on the models involved in the interpretation of the measurement results, i.e., the measurand model and the measurement model. In particular, in all situations in which the random behavior of the input quantities is assumed to be related mostly to the acquisition (or measurement) uncertainty (i.e., when the definitional uncertainty is negligible), the estimator (15) should be applied to improve the quality of information about the measurands before applying $f(\cdot)$.

Conversely, if the input definitional uncertainty prevails on the acquisition uncertainty, the estimator (16) should be applied, to avoid, or at least reduce, the biasing effect that would be introduced by the single application of the nonlinear $f(\cdot)$. The latter consideration has a further implication when the output measurand is obtained by propagating the probability distributions of the input measurands using Monte Carlo simulations, as described in [7]. In fact, in this case the value representing the output measurand is usually estimated by implicitly applying (16) and the expected value is unbiased only if the stochastic behavior of the input quantities is mainly due to their definitional uncertainty, i.e. if the acquisition uncertainty is negligible.

In conclusion, even if the definitional uncertainty and the acquisition uncertainty may affect the input measurands $\boldsymbol{X}$ to a model function $\boldsymbol{Y}=f(\boldsymbol{X})$ in the same way, the preferable method to estimate the measurand value when performing K repeated measurements should be chosen according to the following decisional rules:

→ DECISION RULE 1
If the expression $f(\cdot)$ is a linear (or weakly nonlinear) function, then apply Method 1; else apply decision rule 2.

→ DECISION RULE 2
If the law of propagation of uncertainty may be used (i.e., $f(\cdot)$ is infinitely differentiable at the mean value of $\boldsymbol{X}$) and $\sigma_{def} \approx 0$ (or $\sigma_{def} \ll \sigma_{acq}$), then apply Method 1 (according to the interpretation of Y as having a unique value). Otherwise, if $\sigma_{def} \gg \sigma_{acq}$, then Method 2 should be applied.

APPENDIX A: DERIVATION OF EXPRESSIONS (5)-(12)

Let as refer to N_X as the zero mean random variable with variance σ_X^2 and symmetric (e.g., normal) probability distribution modeling the overall uncertainty in Case A or B. If, in accordance with the example described in Section 3, the estimator (3) is applied to K repeated measured voltage values V_k, then the mean and the variance of the measurand value obtained using Method 1 are given respectively by:

$$E[\hat{P}_{X,1}] = E\left[\left(v + \frac{1}{K}\sum_{k=1}^{K} N_{X_k}\right)^2\right] = v^2 + \frac{\sigma_X^2}{K} \tag{A.1}$$

and:

$$\boldsymbol{var}[\hat{P}_{X,1}] = E\left[\left(v + \frac{1}{K}\sum_{k=1}^{K} N_{X_k}\right)^2\right] - \left(v^2 + \frac{\sigma_X^2}{K}\right)^2 =$$

$$= 4v^2\frac{\sigma_X^2}{K} + \frac{1}{K^3}E\left[N_{X_k}^4\right] + \frac{(2K-3)\sigma_X^4}{K^3} \tag{A.2}$$

where N_{X_k} represents the kth uncertainty contribution. Since in Case A N coincides with the acquisition uncertainty N_{acq}, $\sigma_X^2 = \sigma_{acq}^2$ and $E[N_{X_k}^4] = \gamma_{acq}$. Accordingly, equations (5) and (9) result easily from (A.1) and (A.2).

In Case B, $N = N_{acq} + N_{def}$, where N_{acq} and N_{def} are statistically independent. Thus, $\sigma_X^2 = \sigma_{acq}^2 + \sigma_{def}^2$, $E[N_{X_k}^4] = \gamma_{acq} + 6\sigma_{acq}^2\sigma_{def}^2 + \gamma_{def}$ and the equations (7) and (11) are obtained in the same way as before.

Assume that Method 2 instead of Method 1 is applied to the collected data to estimate the measurand value. In this case, it follows from (4) that its mean value and variance are respectively:

$$E[\hat{P}_{X,2}] = E\left[\frac{1}{K}\sum_{k=1}^{K}(v + N_{X_k})^2\right] = v^2 + \sigma_X^2 \tag{A.3}$$

and:

$$\boldsymbol{var}[\hat{P}_{X,2}] = E\left[\frac{1}{K^2}\left(\sum_{k=1}^{K}(v + N_{X_k})^2\right)^2\right] - (v^2 + \sigma_X^2)^2 =$$

$$= 4v^2\frac{\sigma_X^2}{K} - \frac{\sigma_X^4}{K} + \frac{1}{K}E\left[N_{X_k}^4\right] \tag{A.4}$$

Therefore, by setting $\sigma_X^2 = \sigma_{acq}^2$ and $E[N_{X_k}^4] = \gamma_{acq}$ in Case A and $\sigma_X^2 = \sigma_{acq}^2 + \sigma_{def}^2$ and $E[N_{X_k}^4] = \gamma_{acq} + 6\sigma_{acq}^2\sigma_{def}^2 + \gamma_{def}$ in Case B, the expressions (6), (10), (8) and (12) are finally obtained.

APPENDIX B: THE SIMULATION ALGORITHM

Monte Carlo simulations have been performed by means of a computational engine developed by one of the Authors to model and simulate the evolution of dynamic systems according to the state-variable approach formalized in System Theory. Such a tool [8], [9] enables the qualitative definition of models as directed graphs, whose nodes and arrows represent variables and functional relations between variables respectively. The quantitative definition of each variable is then performed by means of a purely functional language, designed with an explicit reference to MatLab. The simulation graph (shown in Figure 6) producing the results described in Section 5 has been implemented as follows:

- given the chosen number of runs (*runs*=2000 in our examples) and for a given amount of samples K, a matrix (*runs* × K) of Gaussian random samples with

mean equal to the nominal voltage value and variances equal to σ^2_{acq} (Case A) and $\sigma^2_{acq}+\sigma^2_{def}$ (Case B) is generated;
- such a matrix is the basis for the application of the estimators (3) and (4) and for the computation of the corresponding mean and standard deviation values;
- such values are also compared with the theoretical expressions (5)-(8) and (13), (14);
- this procedure is repeated by changing the value of *K* within a preset range.

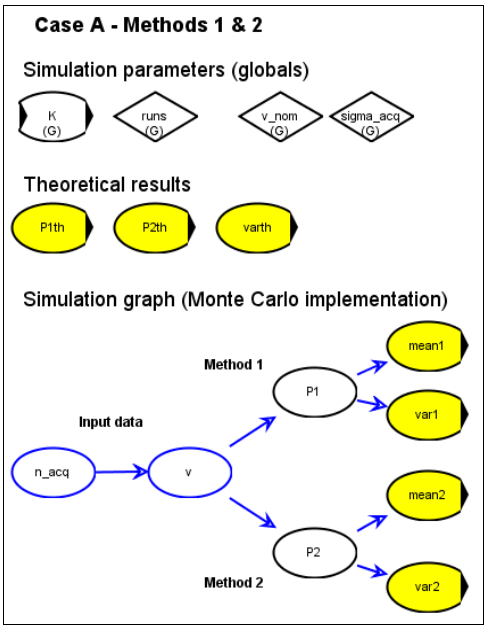

Fig.6 – Simulation graph that implements the Monte Carlo algorithm for Case A (screenshot from the computational engine UI).

The simulation code is as follows.

```
// the system variable 'time' is set so to range
// from 10 to 1000 with a step of 20

runs = 2000          // Number of runs
K = time             // Number of samples
v_nom = 5            // Nominal applied voltage
sigma_acq = 0.5      // Std dev of acquisition noise
P1th = v_nom^2+(sigma_acq^2)/K // Method 1 theoretical
                     //mean value (eq. 5)
P2th = v_nom^2+sigma_acq^2 // Method 2 theoretical
                     // mean value (eq. 6)
varth = 4*v_nom^2*sigma_acq^2/K // Method 1 & 2
                     // theoretical variance (eq. 13)
n_acq = matrix(runs,K,gaussian(0,sigma_acq)) // Matrix
                     // of random noise
v = v_nom+n_acq      // Noisy voltage (*)
P1 = mean(v)^2       // Method 1 mean value (eq. 3) (**)
mean1 = mean(P1)     // Method 1 Monte Carlo mean (***)
var1 = var(P1)       // Method 1 Monte Carlo variance
P2 = mean(v^2)       // Method 2 mean value (eq. 4)
mean2 = mean(P2)     // Method 2 Monte Carlo mean
var2 = var(P2)       // Method 2 Monte Carlo variance
```

where:
(*): The operator '+' is polymorphic; in this case it adds element-by-element a scalar and a matrix.
(**), (***): 'mean' is a dimensional reduction function: if applied to a matrix (case **), it generates the vector of mean values computed over each row; if applied to a vector (case ***), it generates the mean value of its elements.

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