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ERROR ANALYSIS USING FUZZY ARITHMETIC BASED ON T-NORM

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Abstract: The presented algorithm for uncertainty evaluation has three steps: determining the membership function from series of measurements, evaluation of membership function of average value using fuzzy arithmetic based on a t-norm, and addition of systematic errors (based on expert analysis). The parameters of the t-norm are determined by a minimization procedure for a proposed error function.

Keywords: uncertainty, fuzzy intervals, t-norm.

1. INTRODUCTION

Results of measurements are often represented as intervals described as $[x_0 - U(x), x_0 + U(x)]$, where x_0 is a measured value (value of mesurand) and $U(x)$ stands for uncertainty. The interval is estimated using two kinds of data:

1. the series of readings $\{x_n\}_{n=0}^{n=N}$ (raw results of measurement),
2. properties of the measuring instrument and systematic component of uncertainty.

The main problem of error analysis is to find an algorithm for estimating this interval. From algebraic point of view the algorithm is defined by arithmetic, which describes a composition (propagation) of uncertainty components.

The Guide to the Expression of Uncertainty in Measurement [1] recommends two methods of estimation of uncertainty: statistical analysis of measurement series $\{x_n\}_{n=0}^{n=N}$ (based on arithmetic on random variables), and other methods for estimation of the non-random component of error basing on expert knowledge.

In this paper we propose an algorithm based on fuzzy approach to measurement errors described in many papers [13-17]. The present paper is a continuation of our previous paper [2]. Main thesis of our proposition is to replace the arithmetic on random variables by the fuzzy arithmetic based on t-norms [2]. In general we consider the following model of uncertainty:

1. Results of measurements are represented as fuzzy intervals which contain a complete knowledge on measurement results.
2. Propagation of uncertainty is described by means of fuzzy interval arithmetic based on t-norms [4].
3. Fuzzy representation A of the measurement results is equal to the sum:
   $A = A^T_{Av} \oplus_T A_s$
   where A^T_{Av} is a fuzzy number representing the average value and the random component of uncertainty and A_s is a non-fuzzy interval $[-\Delta,+\Delta]$,
4. x_0 such that $A^T_{Av}(x) = 1$ is interpreted as a value of mesurand and for a given $\alpha \in [0,1]$ the α-cut of A is interpreted as uncertainty.

In this paper we propose an algorithm of estimating a fuzzy set corresponding to the measured data series. The proposed algorithm has the following steps:

1) Estimation of a membership function $A_r(x)$ from the empirical measurement raw data $\{x_n\}_{n=0}^{n=N}$ basing on a transformation from the probability distribution function to the possibility membership function (according to [3]).

2) Determining the membership function $A^T_{Av}(x)$ representing the average value of the measured series by means of the arithmetic based on a t-norm [4]. The average value of fuzzy interval is arithmetic mean with addition based on a t-norm.

3) Estimation of the systematic component of uncertainty by expert method which leads to obtaining an interval $A_s(x)$.

4) Addition of both components using arithmetic based on a t-norm in order to obtain a fuzzy interval $A(x)$ representing the inexact result of the measurement.

5) Computation of the uncertainty as a radius of the α-cut of the fuzzy interval $A(x)$.

The algorithm described above requires a previous choice of a t-norm as well as a value of the α-level. The t-norm should be chosen in such a way that empirical data averaging is consistent with averaging based on the t-norm. We propose a procedure for choosing the best-fitted t-norm among t-norms belonging to the given family. The procedure is based on a minimization of the error function $\varepsilon\left(A^{\exp}_{Av}, A^T_{Av}\right)$. The error function is a measure of the divergence between the experimental member function $A^{\exp}_{Av}$ of the average data and A^T_{Av} calculated with use of the tested t-norm. We present the calculation for Yager and Hamacher t-norms.

The problem of choice of the α-level is equivalent to the choice of the confidence level for the estimation of confidence interval. This decision depends on aim of measurement.

The paper is organized as follows. In Chapter 2 we present the model of measurement and uncertainty, in chapter 3 we summarize the representation of measurement in fuzzy intervals with t-norm based arithmetic, and shortly discuss the algebraic representation for composition of uncertainties. In chapter 4 we present some results on probability-possibility transformations and in chapter 5 the results of computations of average values of fuzzy numbers using fuzzy arithmetic. In chapter 6 we propose the method of fitting parameters of t-norms to experimental data. And in Chapter 7 we present the numerical calculations of uncertainty and we discus the results.

2. MODEL OF MEASUREMENT

The measurement is an empirical assignment of numbers (or other mathematical symbols) to properties of objects. From mathematical point of view measurement is described as a homomorphic mapping *f* (measurement mapping) of an empirical relational structure Ω into the mathematical relational structure *Y* (see i.e. [5]):

$$\Omega \xrightarrow[f]{} Y \tag{1}$$

The relational structure *Y* describes results of measurements (values of physical quantities with their structures and uncertainty) and represents the properties of measured objects. In case of the model of an exact measurement *Y* stands for real numbers (real numbers with ordinary order and addition). The results of the inexact measurement are often represented by intervals. Nevertheless it must be underlined that due to a random (or fuzzy) nature of physical phenomena interval representation is only a weak representation of real objects.

The mapping *f* describes the cognition process (with respect to specific properties), therefore *f* must be a homomorphism of structures Ω (empirical) in *Y* (mathematical). In other words *f* must be a representation (*Y*-representation) of empirical structure Ω in mathematical structure *Y*.

In case of measurement of physical quantities (physical attributes) we have to deal with the extensive measurements. We can assume that the empirical structure Ω is endowed with two empirical operations:

1) empirical comparison relation $\prec$ (empirical order),

2) operation $\circ$ of concatenation (composition) of empirical objects.

The empirical operation $\circ$ describes both the empirical additions (concatenation) as well as repetition of experiment and addition (aggregation) of measurement results.

The empirical relation $\prec$ is not in general linear order, but in this paper we do not consider the properties of order relation. Though we would like to emphasis that the comparability depends on the uncertainty.

For purpose of this paper we assume that that the mapping *f* is the representation of empirical operation $\circ$ in mathematical addition $\oplus$ of elements from *Y*:

$$f(\omega_1 \circ \omega_2) = f(\omega_1) \oplus f(\omega_2) \tag{2}$$

In case of the exact measurement we assume that results of measurements *Y* are the real numbers $\Re$ with standard addition + and order <. Such a model is described in the literature very widely (see i.e. [5]). In case of the inexact measurement (measurement with errors) we represent the results of measurement as intervals with standard interval addition $\oplus$ defined as:

$$\overline{a} \oplus \overline{b} = [a_1, a_2] \oplus [b_1, b_2] = [a_1 + b_1, a_2 + b_2] \tag{3}$$

In metrology the interval $\overline{a} = [a_1, a_2]$ is often denoted as $\overline{a} = a_0 \pm \Delta_a$,

where $a_0 = \frac{1}{2}(a_1 + a_2)$ and $\Delta_a = \frac{1}{2}(a_2 - a_1)$.

One can consider the measurement mapping as a pair of mappings (f_1, f_2): $f(\omega) = [a_1, a_2]$ is equivalent to $f_1(\omega) = a_1$ and $f_2(\omega) = a_2$. We can define a different pair of functions (M, U) such that $M(\omega) = a_0$ and $U(\omega) = \Delta_a$

The value $a_0 = M(\omega)$ we interpret as a value of mesurand, and $\Delta_a = U(\omega)$ as uncertainty of a_0.

In fact when considering the reality we think typically about its probabilistic or fuzzy model *F*. Such a model we describe as a representation Φ of real objects Ω in a structure *F*.

$$\Omega \xrightarrow[\Phi]{} F$$

where Φ is a random or fuzzy representation of Ω.

The measurement mapping *f* is a composition of two mappings: Φ representing a model of the reality, and γ which assigns intervals to random variables or fuzzy structures:

$$\Omega \xrightarrow[\Phi]{} F \xrightarrow[\gamma]{} I \tag{5}$$

The existence of a probabilistic representation Φ of physical phenomena is some kind of dogma and we do not try to prove it. On the same principles we assume that fuzzy representation of physical phenomena is adequate to the description of uncertainty and chaotic phenomena.

In case of probabilistic model, the mapping γ assigns a confidence interval for a given confidence level. The uncertainty $U(\omega)$ is given as a radius of the confidence interval:

$$U(\omega) = Rad(\gamma(\Phi(\omega)))$$

where radius of interval [a,b] is given by:

$$Rad([a,b]) = \frac{b-a}{2}$$

In case of the fuzzy model the mapping γ is defined as an α-cut:

$$\gamma(A) = [A]^{\alpha}$$

where *A* is a fuzzy interval.

Error propagation is given by arithmetic in the model *F*:

$$U(\omega_1 \circ \omega_2) = Rad(\gamma(\Phi(\omega_1 \circ \omega_2))) = \\ = Rad(\gamma(\Phi(\omega_1) \oplus_F \Phi(\omega_1))) \tag{6}$$

where $\oplus_F$ is addition in F.

In case of probabilistic models $\oplus_F$ is addition of random variables and in fuzzy models $\oplus_F$ is addition of fuzzy intervals in the arithmetic based on t-norms.

In case of interval representation (when F is an interval structure and γ is an isomorphism) the error propagation has a form:

$$U(\omega_1 \circ \omega_2) = Rad(\Phi(\omega_1) + \Phi(\omega_2))$$

where $\oplus$ is addition of intervals (see equation (3)) and $\Phi(\omega_1), \Phi(\omega_2)$ are intervals.

And due to the fact that

$$Rad([a_1,b_1] \oplus [a_2,b_2]) = Rad([a_1,b_1]) + Rad([a_2,b_2])$$

As a result we get a well-known principle of systematic uncertainty propagation:

$$U(\omega_1 \circ \omega_2) = U(\omega_1) + U(\omega_2). \tag{7}$$

Such an equation is valid only for systematic component of uncertainty. In general addition of random variables or fuzzy sets leads to the reduction of uncertainty. The equation (7) is in fact a definition of a systematic component of uncertainty.

The uncertainty determines a threshold of discrimination of two measured values, hence the uncertainty effects in the existence of incomparability (see.[7]). In case of a principle given by (7) one cannot reduce a measurement error by repeating measurements:

$U(n.\omega) = nU(\omega)$ where $n.\omega$ is n-fold copy of ω (sum of n repetitions of measurement).

This equation one can rewrite as a condition of homotheticity of order: $\omega_1 \prec \omega_1 \Leftrightarrow n.\omega_1 \prec n.\omega_2$.

In case of nonsystematic component of error (the random type) we have: $U(n.\omega) \le nU(\omega)$ and a condition of homothecticity must be replaced by a weaker condition: $\omega_1 \prec \omega_1 \Rightarrow n.\omega_1 \prec n.\omega_2$ (see (11)).

3 FUZZY INTERVALS

Now we recall some definitions about fuzzy sets, t-norms and fuzzy arithmetic (see e.g. [4]). The fuzzy sets on $\Re$ (real numbers) can be understood as a generalization of intervals. We denote fuzzy sets by capital letters $A,B,\ldots$

By $[A]^\alpha$ we denote the α-cut of a fuzzy set A: $[A]^\alpha = \{x \in \Re \mid A(x) \ge \alpha\}$, the kernel of a fuzzy set A is 1-cut: ker(A)=$[A]^1$, and support (0-cut) is defined separately as supp(A)= $[A]^0$ = $[A]^0 = cl(\{x \in \Re \mid A(x) > 0\})$

Among the fuzzy sets we distinguish the class of *fuzzy intervals FI*: $A \in FI$ if $[A]^1 \neq \emptyset$ and $[A]^\alpha$ is a closed bounded interval for all $\alpha \in [0,1]$. In this paper we assume that results of measurements are represented by fuzzy intervals.

In the class of fuzzy intervals we distinguish two classes of fuzzy intervals: *FIS and FIR.*

FIS are the fuzzy intervals whose membership functions are the characteristic functions of the closed bounded intervals i.e.: $A \in$ *FIS* if supp[A]=ker[A]. We assume that in our model of uncertainty *FIS* represent systematic error and we call them **fuzzy systematic interval**.

FIR are the fuzzy numbers with kernel consisting of only point: ker[A] = $\{a_0\}$. In our model we assume that fuzzy numbers represent the measurement results with random errors only, and we call them **fuzzy nonsystematic intervals**.

Arithmetic of fuzzy intervals is defined as the extension of interval arithmetic, applying the Zadeh extension principle with a t-norm T [4]:

$$(A \oplus_T B)(z) = \sup_{x+y=z} T(A(x), B(y)), \quad \text{for } z,x,y \in \Re \tag{8}$$

Using the Fuller theorem [10] one can write:

$$[A \oplus_T B]^\alpha = \bigcup_{T(\xi,\eta) \ge \alpha} ([A]^\xi \oplus [B]^\eta) \tag{9}$$

where $\oplus$ denotes the addition (3) of intervals $[a_1,a_2] \in I$.

This equation is very convenient for numerical calculations of the sum of the experimental membership functions.

The structure $\langle FIS, \oplus_T \rangle$ is isomorphic to the interval structure $\langle I, \oplus \rangle$; therefore the interval addition (3) describes the propagation of systematic errors as in (7).

In the interval structure $\langle I, \oplus \rangle$ we can define the interval order and in paper [6] the problem of interval representation of empirical structure Ω is solved in case of homothetic order (the condition of homotheticity is in our case gives the properties (7)). In general we can construct only weak interval representation (10).

Each fuzzy interval can be decomposed uniquely into two parts: systematic and nonsystematic type [11]:

$A = A_r \oplus_T A_s$ where $A_r \in FIR$ and $A_s \in FIS$,

assuming that A_s is isomorphic with a non-fuzzy interval of the type $[-\Delta, +\Delta]$, for some $\Delta \ge 0$. This decomposition expresses fact that uncertainty consists of two components: random and systematic.

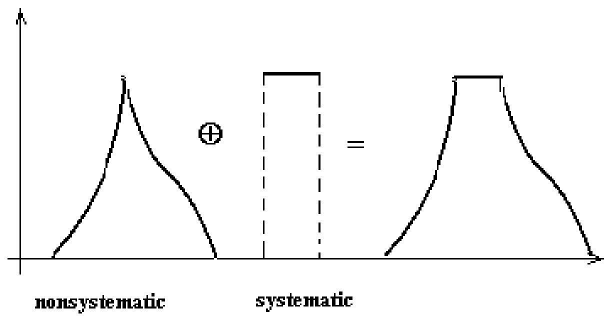

Fig 1 Graphical presentation of fuzzy interval decomposition

The random component has (from definition) the property that the uncertainties decrease while averaging data series of measurement:

$$\lim_{N\to\infty}\frac{1}{N}U\left(\sum_{n=1}^{N}\omega_n\right)=0 \tag{11}$$

where: $\omega_n \in \Omega$, the sum:

$$\sum \omega_n = \omega_1 \circ \ldots \circ \omega_N$$

is a composition of empirical objects.

This equation is represented in fuzzy model by (see [2] and [8]):

$$\lim_{N\to\infty}\frac{1}{N}\bigoplus_{n=1}^{N} A_n = A_s \text{ where } A_s \in FIS. \tag{12}$$

The sum $\bigoplus_{n=1}^{N} A_n = A_1 \oplus_T \ldots \oplus_T A_N$ is a t-norm based arithmetic sum.

The equation (12) has a natural interpretation that the random component of uncertainty converges to zero if a number of averaging grows to infinity.

In the structure of fuzzy intervals we can introduce the metrics for evaluation of a divergence measure. In literature one can find many concepts of divergence measure [9], but we chose the max norm:

$$\rho(A,B)=\max_{\alpha\in[0,1]}\rho_I\left([A]^\alpha,[B]^\alpha\right) \tag{13}$$

where ρ_I is the measure of distance between intervals given by

$$\rho([a_1,a_2],[b_1,b_2])=\max\left(|a_1-b_1|,|a_2-b_2|\right).$$

We use formula (12) for computation of the difference between empirical and theoretical fuzzy intervals in order to obtain the best-fitted t-norm.

4. ESTIMATION OF MEMBERSHIP FUNCTION

Probability has a simple frequency interpretation and therefore the algorithm for estimating a probability distribution function for empirical data is based on a construction of histogram. In order to obtain a membership function we use two methods:

1. Expert methods based on physical properties of instruments for determination of systematic error.
2. Analysis of data series using similar to probabilistic methods. These methods based on probability-possibility transformation [3].

Among the several methods we choose the method basing on an assumption that the confidence interval is equivalent to the α-cut of a fuzzy interval A corresponding to the random variable:

$$[A]^\alpha=\left[F^{-1}\left(\frac{\alpha}{2}\right),F^{-1}\left(1-\frac{\alpha}{2}\right)\right] \tag{14}$$

where F is a cumulative distribution function: $F(x)=\int_{-\infty}^{x} f(x')dx'$,

where f is probability density function.

The equation (14) can be implemented with ease for obtaining the estimation of the membership function corresponding to the empirical measurement series $\{x_n\}_{n=1}^{N}$. On figure 2 we show the experimental histogram (measurements of noised voltage signal) and the membership function obtained.

One of the basic features of the algorithm (14) is that it converts a flat pick into a sharp pick (for uniform distribution we obtain the triangular membership function), but a tail is preserved (and smoothed).

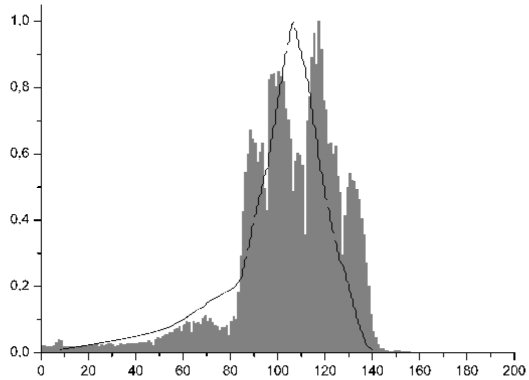

Fig. 2. The histogram and obtained membership function (solid line) for empirical data (signal with multi-modal noise) in arbitrary units

5. STATISTICS WITH T-NORM.

Frequently used statistics are arithmetic means (average value) from experimental series. The average value is a good estimator of the expected value of random variable and typically this statistics is used as a mesurand.

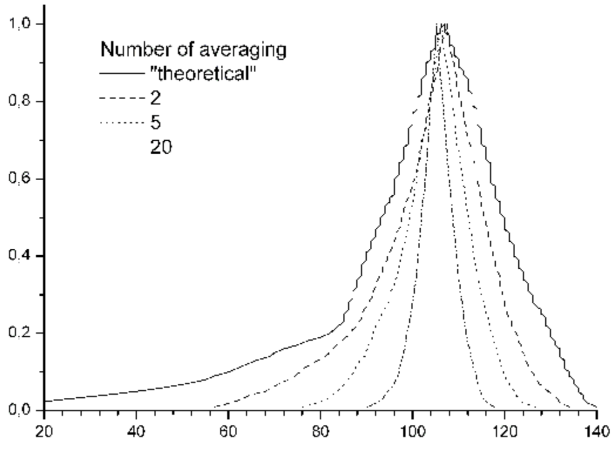

Fig. 3.a. The membership function for averaging empirical data from Fig1. (theoretical line –from all measured data) for 2, 5 and 20 numbers of averaging points.

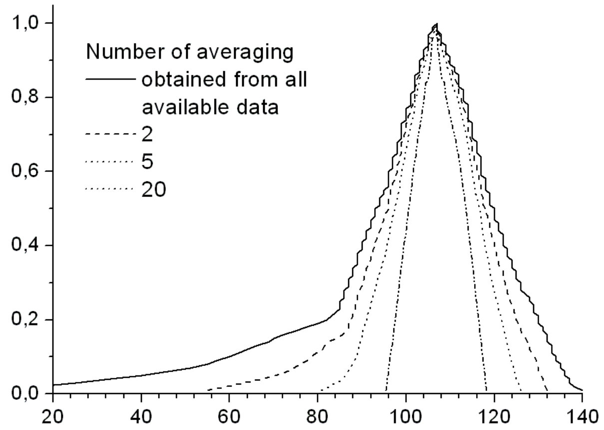

Fig 3.b. The Averaged membership function as a function of number of averaging N, for measured noice and for Yager t-norm with p=3.

The figure3.a shows narrowing of the membership functions obtained in the procedure of subsequent averaging. The membership function is computed using (14) for series of data presented on Fig.2. (all computation are made for 100 α-levels).

Similarly, figure 3.b presents averaged membership functions for various numbers of averaging for empirical data for Yager t-norm with $p = 3$ (denoted as Y(3)).

For fuzzy intervals we use the average value of membership function in arithmetic based on t-norm:

$$A_{Av,N}^{T} = \frac{1}{N} \mathop{\oplus_T}_{n=1}^{N} A \tag{15}$$

where A is membership function obtain from experimental data by transformation(14). The summation $\oplus_T$ is a t-norm based sum.

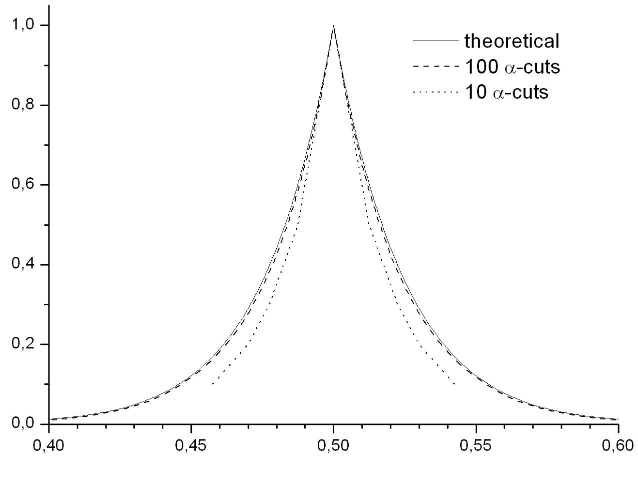

Fig. 4. Comparison of theoretical (solid line) and numerical membership function for 20 averages for 10 and 100 αcurs

In order to estimate a t-norm which leads to the narrowing as in the figure 3 we need to calculate sums of a few membership functions basing on a chosen t-norm. Due to the discrete values of the membership functions obtained as a result of the numerical procedure it is necessary to examine the difference between numerical and analytical summation. Numerical summation is carried out by means of the equation (9) for a discrete set of α values.

Figure 4 presents a comparison between numerical and analytical profiles of membership functions for averaging of twenty symmetric triangular fuzzy numbers with support equals to [0, 1] basing on a product t-norm depending on the number of α-cuts. Similar results obtained for the Y(2) t-norm show that for 100 α-cuts a maximal difference between the profiles is less than $5 \cdot 10^{-3}$.

Using Fuller's theorem it is easy to estimate the error of fuzzy sets addition by means of the discretization procedure. In case of symmetric triangular fuzzy numbers with support equals to [0, 1] a number $N > 2/\varepsilon$ of α-cuts assures that the error in the sense of (13) is less than ε. In case of the fuzzy number coming from normal distribution for Yager t-norm with $p = 2$ and 20 averaging, a number of α-cuts which guarantees that the error of 0.05-cut is biased with the error less than 0.01 is about 100.

In the figure 3a we can observe a peak shift if a number of averaging grows. The shift in 3a is an effect of asymmetry of the data distribution and due to fact that the arithmetic mean converges to the expected value, which in this case is lower then median.

The same shift can be observed in fig.6 where the membership function of averaged data has the pick at the point 105 in contrary to the pick position computed with t-norm (the pick is at median).

6. EXPERIMENTAL EXAMINATION OF T-NORM

In order to obtain a t-norm belonging to the given one-parameter family, which fits the best in the sense of equation (9) we carried out the procedure described in chapter one.

The following figures present results of calculations carried out for Yager and Hamacher t-norms families.

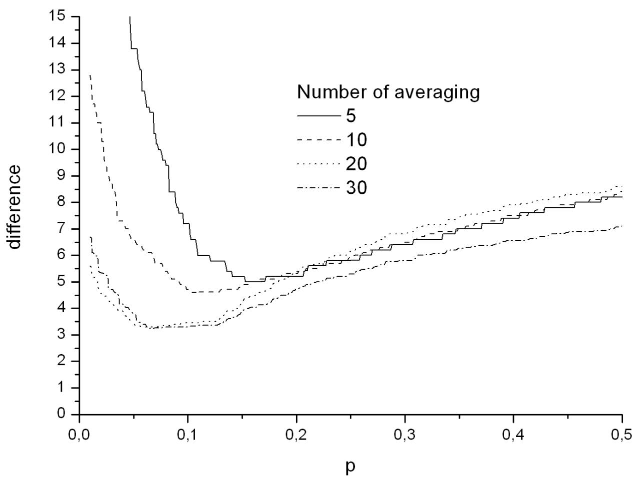

Fig.5. A difference (13) between membership functions of averaged empirical data (fig. 2) and Hamacher t-norm as a function of parameter *p*.

Figure 5 shows values of difference between numerical and theoretical averaging in the sense of equation (13) as a function of p parameter indexing Hamacher t-norms. Figure 6 presents examples of the empirical membership function for averaging of 20 elements and calculated functions for a various values of *p*. Each series of data contains 1000 numbers.

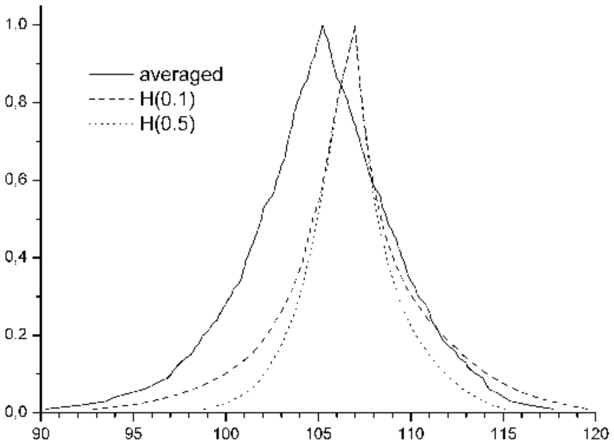

Fig.6. The membership functions or averaged and calculated data with Hamaher t-norm, for measured noised signal (for averaging of 20 components).

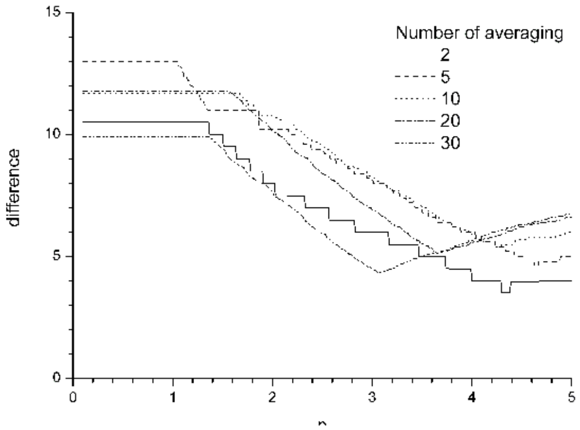

Fig.7. The difference between membership functions of averaged empirical data and Yager t-norm as a function of parameter p.

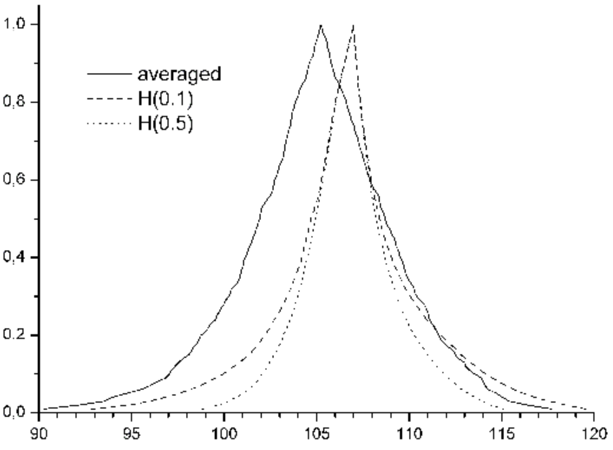

Fig..8. The averaged (for N=20) and calculated membership functions for measured noised signal.

Similarly, the following figures present results for random generated normally distributed series of numbers.

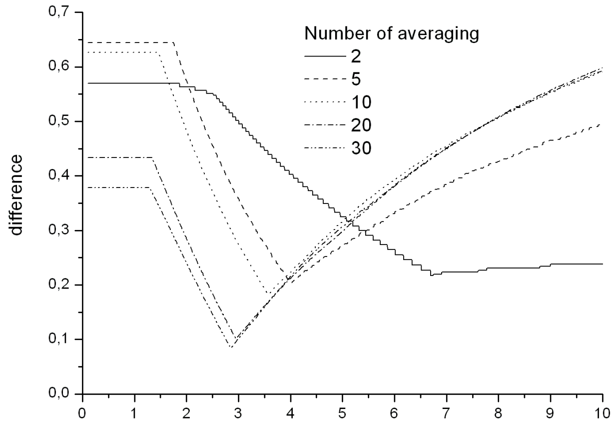

Fig.9. The difference (13) between membership functions of averaged random generated normal distribution and Yager t-norm as a function of parameter *p*.

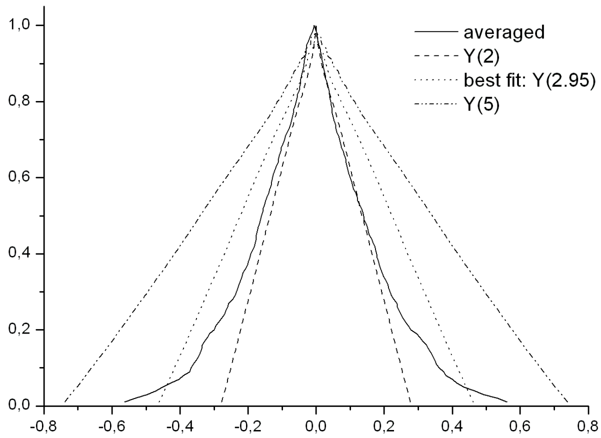

Fig..10. The averaged and calculated with Yager t-norm membership functions for random generated normally distributed signal for 20 averages.

The differences presented in the figures 5, 7 and 9 depend on a number of averaging, hence we can observe some minima. As far as applications are concerned the minima for 20-30-fold averaging are more interesting and we think that the stabilization of minima position is not a numerical effect in spite of the fact that errors of the minima positions might be comparable with depth of the minima. This problem requires farther studies.

6. CONCLUSION

The work presents explicitly application of fuzzy numbers arithmetic based on t-norms in data analysis and shows concrete numerical results. It allows drawing conclusions referring to possibilities, limitations, and expectations of such an approach. The uncertainty evaluated as a radius of the α-cut decreases as a number of averaging grows in a similar way as in case of empirical averaging. On figure 11 we show the results of computation of uncertainty for empirical data as the effect of empirical averaging and averaging with use of Yager t-norm for $p = 3$ and Hamacher for $p = 0.1$ Moreover we present statistical confidence interval calculated according to GUM [1] as:

$$U(x) = K_{0.95} s$$

where s is the standard deviation estimated from measured data and $K_{0.95}$ is the coefficient of extension. Finally "theoretic confidence interval" presents a radius of confidence interval of the distribution of mean.

In figure 11 we may observe differences between uncertainty estimated by means of averaging with Yager and Hamacher t-norms. On the other hand statistical methods give results very similar to the results obtained for the Hamacher t-norm. We suspect that it is always possible to choose a t-norm which gives results very similar to the results obtained by statistical methods.

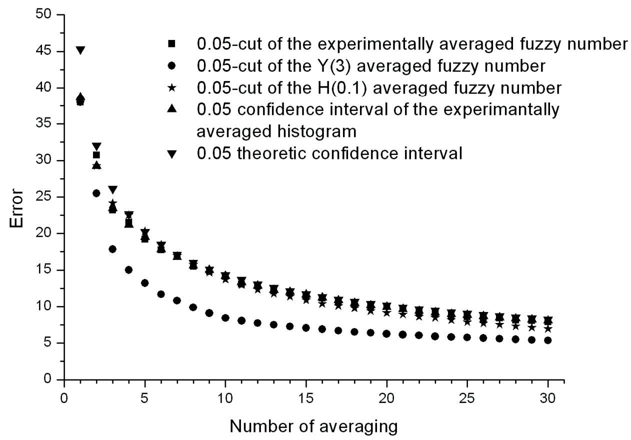

Fig.11 Uncertainty vs number of averaging for empirical data calculated from empirical averaging,
and with Yager and Hamacher t-norms.

The algorithm used for obtaining a fuzzy membership function from empirical distribution [3] converts a confidence interval into the α-cut interval. The algorithm smoothes the noise and sharpens flat regions of the distribution as well as convert concave probabilistic distribution functions into convex membership functions. Despite its drawbacks, the algorithm seems very appropriate for low α values.

On the other hand, the algorithm of numerical averaging basing on Fuller's theorem it seem to be quite efficient and leads to small errors when number of α-cuts is large enough in comparison to the number of averaging. The further study on properties of numerical procedure is necessary.

The average value seems not to be optimal statistics for asymmetric distributions and studying different statistics might be beneficial. The problem is when distribution is asymmetric arithmetic because the mean from experimental data converges to excepted value, which (for asymmetric distributions) is not equal to median.

Moreover the examined Yager and Hamacher t-norms do not lead to the good agreement with experimental averaging neither for noised signal nor for random generated numbers. It may be necessary to examine more families of t-norms.

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