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EDUCATIONAL METHODS AS MATHEMATICAL AND PHYSICAL APPROACHES DEMONSTRATED WITH TYPICAL EXAMPLES

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Abstract: Three typical examples with great relevance to practice are demonstrating that as well mathematics as physics must be used to solve problems and especially to gain closed solutions as necessary for application in practice. Therefore both methods have to be teached in measurement education also to-day.

Keywords: educational methods, approaches, typical examples

1. INTRODUCTION

While in the era of the slide-rule physical considerations were necessary to gain the order of magnitude of the solution today the computer yields the complete solution. Nevertheless also to-day the physical approach is necessary to gain an approximate result, which is needed for three reasons: Firstly to check the solution and by the way to avoid errors in programming, secondly often the exact solution in practice because of the variations of the parameters is not necessary and thirdly a closed solution in the form of a not too expendable formula facilitates the application in practice.
The paper deals with these methods and shows with three typical examples the importance of these approaches to solve problems of measurement science.

2. ERRORS IN FREQUENCY MEASUREMENT

In electronic measurement and communication systems with time-dependent parameters so-called rheolinear systems appear. Due to a theorem of Floquet in the solutions of the Hill- and Mathieu-differential equation a characteristic exponent μ occurs and thus shows the both desirable or undesirable behaviour of the system as synchronization or instabilities. To gain μ energy and phase investigations are used and practical applications are given:
The first example may be frequency measurement using a reference frequency. In literature only two frequency domains are distinguished: Inside the synchronisation domain no difference frequency exists, while outside the difference frequency is assumed to be $\Omega_1 - \Omega_2$. In reality because of the inevitable coupling of the two generators an error between the ideal difference frequency and the real difference frequency appears.
Rheolinear systems may be described by means of the general differential equation

$$y``(t) + a_1(t)\, y`(t) + a_2(t)\, y(t) = x(t) \qquad (1)$$

where $a_1(t)$ physically means the variation of the damping and $a_2(t)$ the variation of the Eigenvalue and thus the resonance frequency.
In this problem we are especially interested in the Eigenvalues. Therefore the solution of the homogenous *Hill* equation is sufficient

$$z(t)`` + \Phi(t)\, z(t) = 0 \qquad (2)$$

Due to a theorem of *Floquet* [1] the following solutions are existing

$$z(t) = e^{\mu t} f(t) + e^{-\mu t} g(t) \qquad (3)$$

The mathematical treatment is very difficult even for the special case of a periodic $\Phi(t)$, the so-called *Mathieu* differential equation,

$$z(t)`` + \Omega_0^2(1+\sigma \sin\omega t)\, z(t) = 0 \qquad (4)$$

and therefore the solutions thus gained are not suitable for use in practice [2].
The solutions are stable or instable depending on the real part of the characteristic exponent μ of equation (3). In detail the Diagram of *Ince* and *Strutt* as shown in Figure 1 demonstrates these stable and instable regions [2].
To get results of the behaviour of the system and especially of the characteristic exponent μ we will use not the mathematical approach but investigations based on physical considerations and approximations as in principle described in [3].

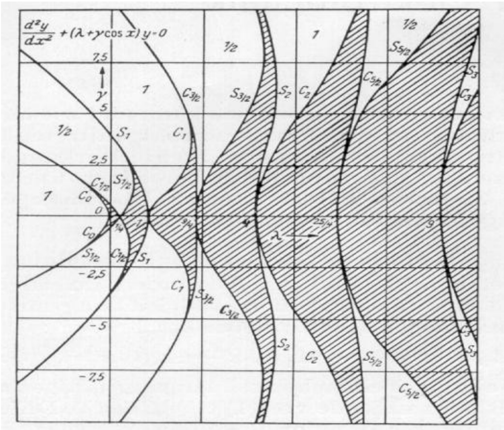

Fig. 1. Diagram of *Ince* and *Strutt*

To get results of the behaviour of the system and especially of the characteristic exponent μ we will use not the mathematical approach but investigations based on physical considerations and approximations as in principle described in [3].
We start with equation (4), the so-called *Mathieu*-equation describing for instance an oscillator with time-varying capacitance with $\sigma = \Delta C/2C_0$ as shown in Figure 2.

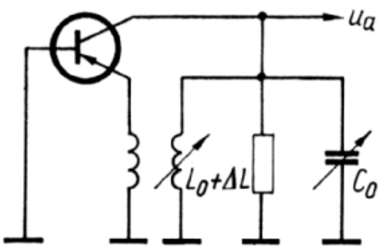

Fig. 2. Oscillator with time varying parameters

As Figure 3 demonstrates it depends of the phase φ if there will be generated an additional effective capacitance or a negative damping leading to the characteristic exponent μ. Especially for $\varphi = \pi/2$ the capacitance always will be reduced while the charge is great and vice versa. Thus energy will be "pumped" into the circuit leading to a negative damping μ.. In the higher instable ranges the harmonics are generating the same effect.
Using this method the characteristic exponent may be gained [5;6] leading to the relation

$$\mu = \sqrt{\Delta\Omega_0{}^2/4 - \overline{\Delta\Omega^2}} \qquad (5)$$

with the locking range $\Delta\Omega_0$ and the deviation to the resonance frequency $\Delta\Omega$..

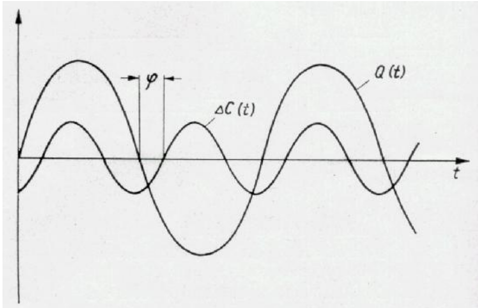

Fig. 3. Charge Q and time-varying capacitance ΔC

Figure 4 shows the course of μ. Inside the synchronisation range $\Delta\Omega_o$ the exponent μ is real, that means a negative damping and not a frequency difference (synchronization). Outside this range μ is imaginary, that means a frequency deviation (error) in comparison with the ideal difference frequency $\Delta\Omega = \Omega_1 - \Omega_0$.
In the case of frequency measurement using a reference frequency as shown in Figure 5 because of the inevitable coupling of the two generators the problem is as investigated before.

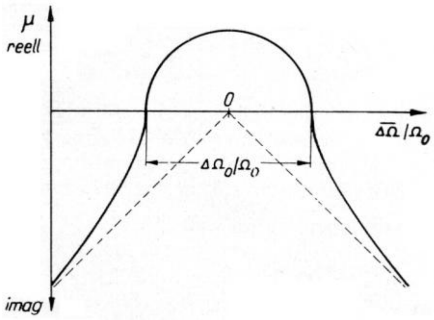

Fig.. 4. Course of the characteristic exponent

To gain the error between the ideal difference frequency $\Delta\Omega$ and the real difference frequency we use equation (5) . So the real difference frequency follows

$$\sqrt{\Delta\Omega^2-\Delta\Omega_o^2/4}=\Delta\Omega[1-0,5(\Delta\Omega_o^2/4\Delta\Omega^2)] \qquad (6)$$

That means: If for instance the difference frequency is by the factor 100 greater than the synchronization range $\Delta\Omega_0$ the relative error still runs to $0,125 \cdot 10^{-4}$! In practice the permissible error in high precision frequency or time measurement may be less than 10^{-10} to 10^{-12} [7]. From this fact it follows that the synchronization range has to be smaller than $0,3{*}10^{-4}$ to $0,3{*}10^{-5}$ of the difference frequency. Finally it may be hinted that capacitive or inductive sensors using the principle of frequency modulation prove the same behaviour. The string-extensometer also belongs to this group [6, 8].

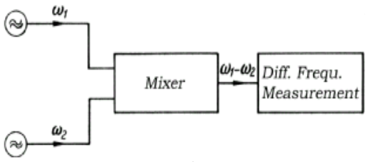

Fig. 5. Frequency measurement

3. ERRORS IN DYNAMIC MEASUREMENTS

To show the typical approach and to demonstrate the importance of estimation methods in education here as another example the answer of a system to a pulse function may be treated [8,9]. To estimate dynamic errors in first approximation instead of the real step answer – an e-function - a ramp function with the transient time T_T and as an input a pulse-shaped function with the width ΔT is used as shown in Fig.6.

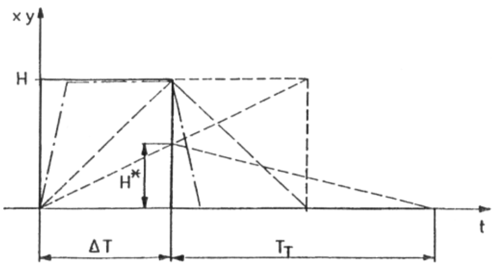

Fig. 6. Answer to a pulse function (estimation)

Now three typical cases may be distinguished:

a) $T_T = \Delta T$ Here the pulse is distorted but the pulse height is still correct (dashed curve).

b) $T_T > \Delta T$ This leads to an error of the pulse height (short dashes).

c) $T_T < \Delta T$ The pulse shape can be represented (chain curve).

The investigations show that a heavily prolonged trailing edge means a dynamic error - as especially practicians with a lot of experience know. On the other hand it is possible to estimate the error using the relation of the flank sides of the output signal: The real height of the input signal H is - using the symbols of figure 6 - thus correcting the measured wrong height H*

$$H \cong H^* \, T_T / \Delta T \qquad (7)$$

To explain the results in the field of measurement it is remarkable that minor and medium errors in amplitude height frequently have more detrimental effects in practice than very large measuring errors. For small errors the device or element that has been dimensioned according to this measurement will function for a certain time due to the safety margins and possibly withstand the initial tests. After have been produced in series and operated for a certain time, however all elements fail according to the fatigue curve for the number of stress reversals possible up to the failure. On the other hand major measuring errors become evident during testing.

There are direct parallelisms to the quality of our students:

If they are intelligent and diligent – no problem. If they are intelligent but lazy, also no problem, because they will do less but they don´t make mistakes. Even if they are stupid and lazy its not a problem, because everybody will expect they make errors. Dangerous are those which are stupid and diligent, because they produce not so extreme and not expected errors.

4. ESTIMATION OF THE SOLUTION OF DIFFERENTIAL EQUATIONS AND OF THE TRANSIENT TIME

In linear systems the behaviour is described by a linear differential equation of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 \ddot{y} + a_1 \dot{y} + a_0 y = b_1 x + b_2 \dot{x} + \cdots + b_{m-1} x^{(m-1)} + b_m x^{(m)} \qquad (8)$$

Normalization leads to the form

$$T_n^{\,n} y^n + \cdots T_2^{\,2} \ddot{y} + T_1 \dot{y} + y = (b_1/a_0)\, x + (b2/a_0)\, \dot{x} + \cdots + (b_m/a_0)\, x^{(m)} \qquad (9)$$

Comparison of both equations taking into consideration dimensions shows that

$$T_r^{\,r} = a_r/a_0 \qquad (10)$$

are time-constants, while

$$\Delta y/\Delta x = b_1/a_0 \qquad (11)$$

means the static sensitivity of the system.

In the exact solution the eigenvalues are the zeros of the n-dimensional characteristic function

$$T_n^{\,n} y^n + \cdots T_2^{\,2} \ddot{y} + T_1 \dot{y} + y = 0 \qquad (12)$$

A first appproximation uses instead of these eigenvalues the time-constants $T_r^{\,r}$ of Equ. (10). Due to the fact, that

$$e^{-3} = 1/20 \qquad (13)$$

that means 5%, the transient-time may be estimated to be three-times the value of the greatest time-constant $T_r^{\,r}$. This estimation will be the better, the greater the differences between the time-constants are. (7)

5. CONCLUSION

The paper deals with important educational methods as mathematical and physical approach. Three typical examples are treated to show that also to-day both methods are important to gain an approximate result necessary to avoid errors in check solutions of computer analysis and to avoid errors in programming. Often only the physical approach leads to a closed solution as advantageous for application in practice.

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