Varying Risk Confidence

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Source dependence

"... the utility of money or other rewards is not independent of the circumstances under which it is obtained ..." (Smith, 1969 on Ellsberg, 1961)

"... risk-taking is indeed content-specific ..." (Weber, Blais and Betz, 2002)

"... we propose to clarify, with respect to clients' risk-bearing capacity, that any particular consumer has not only one and overall risk attitude but different risk attitudes towards different investment targets ... " (European Securities and Markets Authority, May 2014)

Sources of risk can differ for several reasons

- knowledge of the stochastic nature of the risk being faced
- moral implications
- social implications
- time frames

OUR OBJECTIVE IS TO PROVIDE A GENERAL FRAMEWORK FOR DECISION THEORY UNDER UNCERTAINTY WITH MULTIPLE SOURCES

Earlybird example

A paradigmatic example is given by the choice between two cars

Assume available cars **X** and **Y** are described by only two characteristics: gas mileage and safety but different specialistic journals report different values for these characteristics, e.g.

	Gas Mileage		Safety	
X	A	20	A	5
	B	21	B	4.5
	C	17	C	4
	D	19	D	5
Y	A	24	A	4
	B	25	B	4
	C	23	C	3.5
	D	22	D	4.5

The end at the beginning

$$U\left(X_1,...,X_n\right)=u_1^{-1}\mathbb{E}_{p_1}\left[u_1\left(X_1\right)\right]+...+u_n^{-1}\mathbb{E}_{p_n}\left[u_n\left(X_n\right)\right]$$

where $\left(X_1,...,X_n\right)$ is a vector of omogeneous random attributes and $\left(u_1,...,u_n\right)$ are the corresponding utilities.
Therefore, $u_i^{-1}\mathbb{E}_{p_i}\left[u_i\left(X_i\right)\right]$ is the certainty equivalent of X_i w.r.t. u_i and p_i, i.e., the sure quantity which is equivalent to X_i

This can be extended to the case in which attributes are not homogeneous

$$U\left(X_1,...,X_n\right)=W_1\left(u_1^{-1}\mathbb{E}_{p_1}\left[u_1\left(X_1\right)\right]\right)+...+W_n\left(u_n^{-1}\mathbb{E}_{p_n}\left[u_n\left(X_n\right)\right]\right)$$

(but not today, except...)

Ellsberg example

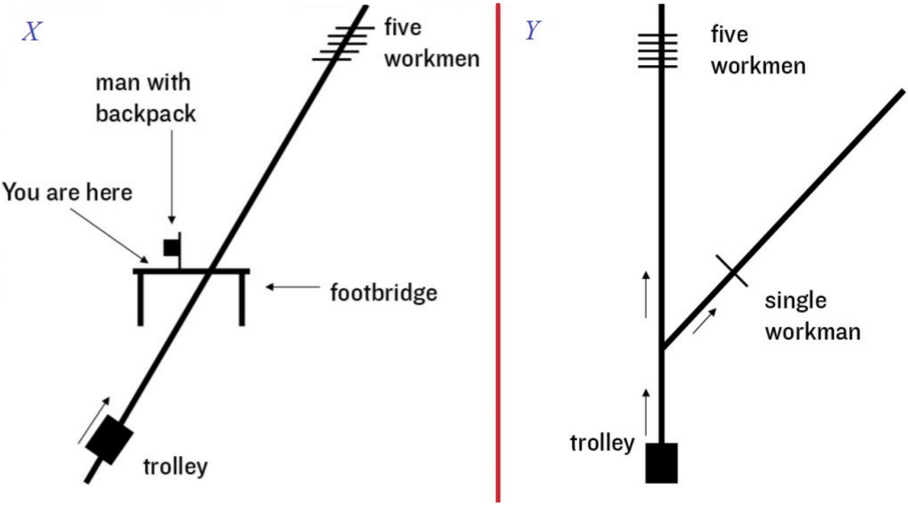
"... You have the following information. Urn 1 contains 100 red and black balls, but in a ratio entirely unknown to you; there may be from 0 to 100 red balls. In Urn 2, you confirm that there are exactly 50 red and 50 black balls ...

1. Which do you prefer to bet on, r_1 or b_1; or are you indifferent? That is, drawing a ball from Urn 1, on which "event" do you prefer the $100 stake, red or black: or do you care? ..."

and the typical preference pattern is

$$\text{red on Urn 1} \sim \text{black on Urn 1} \prec \text{red on Urn 2} \sim \text{black on Urn 2}$$

Moral dilemmas (inspired by Thomson, 1976)



In both cases, 1 casuality with probability p and 5 with probability $1 - p$

Setup

- The multiplicity of sources is represented by a family $\{S_i\}_{i \in I}$ of finite state spaces
- Choice consequences are real numbers (e.g., monetary rewards or casualities)
- A prospect depending on source i is a random variable $X_i : S_i \longrightarrow \mathbb{R}$ where $x = X_i(s_i)$ is the consequence of choosing X_i in state s_i
- The set of all prospects depending on source i is denoted by $\mathcal{X}_i$
- Preferences are represented by a binary relation $\succsim$ on $\mathcal{X} = \bigcup_{i \in I} \mathcal{X}_i$
- $\succsim_i$ denotes the restriction of $\succsim$ to $\mathcal{X}_i$

Ellsberg example again

Setting $S_1 = \{r_1, b_1\}$ and $S_2 = \{r_2, b_2\}$ bets have the form

	r_1	b_1
R_1	100	0
B_1	0	100

and

	r_2	b_2
R_2	100	0
B_2	0	100

and the typical preference pattern is

$$R_1 \sim B_1 \prec R_2 \sim B_2$$

Source dependent expected utility

Axiom (SDEU)

For every $i \in I$, $\succsim_i$ on $\mathcal{X}_i$ is a continuous and strictly increasing weak order satisfying cardinal coordinate independence.

Lemma (Wakker, 1988)

A binary relation on $\mathcal{X}$ satisfies SDEU iff for every $i \in I$, there exist a strictly increasing and continuous function $u_i : \mathbb{R} \to \mathbb{R}$ and a strictly positive probability measure $p_i \in \Delta(S_i)$ such that, given $X_i, Y_i \in \mathcal{X}_i$,

$$X_i \succsim_i Y_i \iff \mathbb{E}_{p_i}[u_i(X_i)] \geq \mathbb{E}_{p_i}[u_i(Y_i)].$$

Moreover, the elements of $\{u_i\}_{i \in I}$ are unique up to positive affine transformations and the elements of $\{p_i\}_{i \in I}$ are unique.

Transitivity

Axiom (TRAN)

The relation $\succsim$ is transitive.

Comparability

Axiom (COMP)

For every $i, j \in I$ and $x \in C$, there is $y \in C$ such that $x\mathbb{I}_{S_i} \sim y\mathbb{I}_{S_j}$.

For example, if the DM is a risk manager

- for *financial risks*, a state s_i describes the situation of financial markets
- for *operational risks*, a state s_j describes the situation of internal processes (people and systems behavior) and external accidents

For the DM a sure \$ 10 million financial loss might not be equivalent to a sure operational loss of the same amount, but, e.g., it could be equivalent to a sure \$ 12 million operational loss

Basic representation

Proposition

Let $\succsim$ be a binary relation on $\mathcal{X}$. The following conditions are equivalent:

(i) *$\succsim$ satisfies SDEU, TRAN, and COMP;*

(ii) *there exist three families*

- $\{u_i\}_{i \in I}$ *of strictly increasing continuous functions* $u_i : \mathbb{R} \to \mathbb{R}$,
- $\{p_i\}_{i \in I}$ *of strictly positive probability measures* $p_i \in \Delta(S_i)$,
- $\{\delta_{ji}\}_{j,i \in I}$ *of strictly increasing continuous functions* $\delta_{ji} : \mathbb{R} \to \mathbb{R}$ *satisfying* $\delta_{ki} = \delta_{kj} \circ \delta_{ji}$ *and* $\delta_{ii} = \mathrm{id}_{\mathbb{R}}$ *for all* $i, j, k \in I$,

such that, given $i, j \in I$, $X_i \in \mathcal{X}_i$ *and* $Y_j \in \mathcal{X}_j$,

$$X_i \succsim Y_j \iff u_i^{-1}\mathbb{E}_{p_i}[u_i(X_i)] \geq \delta_{ij}\left(u_j^{-1}\mathbb{E}_{p_j}[u_j(Y_j)]\right).$$

The elements of $\{u_i\}_{i \in I}$ *are unique up to positive affine transformations, the elements of* $\{p_i\}_{i \in I}$ *and* $\{\delta_{ji}\}_{j,i \in I}$ *are unique.*

Inter-source rate of substitution

Notice that for every $i, j \in I$, $x, y \in \mathbb{R}$,

$$x\mathbb{I}_{S_i} \sim y\mathbb{I}_{S_j} \iff y = \delta_{ji}(x)$$

If the only difference between source i and source j is the quality of the information about the stochastic nature of state realizations (like in the Ellsberg example), it is compelling to consider

Axiom (SCOM)

For every $i, j \in I$ and $x \in \mathbb{R}$, $x\mathbb{I}_{S_i} \sim x\mathbb{I}_{S_j}$.

Which strengthens COMP and implies $\delta_{ij} = \mathrm{id}_{\mathbb{R}}$ for all $i, j \in I$

Stochastically different sources

Proposition

Let $\succsim$ be a binary relation on $\mathcal{X}$. The following conditions are equivalent:

(i) *$\succsim$ satisfies SDEU, TRAN, and SCOM;*

(ii) *there exist two families*

- $\{u_i\}_{i \in I}$ *of strictly increasing continuous functions* $u_i : \mathbb{R} \to \mathbb{R}$,
- $\{p_i\}_{i \in I}$ *of strictly positive probability measures* $p_i \in \Delta(S_i)$,

such that, given $i, j \in I$, $X_i \in \mathcal{X}_i$ *and* $Y_j \in \mathcal{X}_j$,

$$X_i \succsim Y_j \iff u_i^{-1} \mathbb{E}_{p_i} [u_i (X_i)] \geq u_j^{-1} \mathbb{E}_{p_j} [u_j (Y_j)] .$$

The elements of $\{u_i\}_{i \in I}$ *are unique up to positive affine transformations, the elements of* $\{p_i\}_{i \in I}$ *are unique.*

A MAUT approach

Assume $I = \{1, ..., N\}$ and set $\mathcal{X}^* = \mathcal{X}_1 \times ... \times \mathcal{X}_N$ and preference $\succsim^*$ are defined on $\mathcal{X}^*$

$$\mathbf{X} = (X_1, X_2, ..., X_n)$$

is a "portfolio of risks" depending on different sources
Identify

$$\begin{aligned} \mathcal{X}_i &\equiv \left\{(X_1, X_2, ..., X_n) \in X^* : X_j = 0\mathbb{I}_{S_j} \text{ if } j \neq i\right\} \\ \mathcal{X} &\equiv \bigcup_{i \in I} \left\{(X_1, X_2, ..., X_n) \in X^* : X_j = 0\mathbb{I}_{S_j} \text{ if } j \neq i\right\} \end{aligned}$$

and denote by

- $\succsim$ the restriction of $\succsim^*$ to $\mathcal{X}$
- $\succsim_i$ the restriction of $\succsim$ (i.e. of $\succsim^*$) to $\mathcal{X}_i$

Monotonicity, continuity, and independence

Axiom (MONO)

If $X_i \succsim_i Y_i$ *for all* $i \in I$, *then* $(X_1, X_2, ..., X_n) \succsim^* (Y_1, Y_2, ..., Y_n)$.

Axiom (CONT)

$\succsim^*$ *is continuous.*

Axiom (INDI)

If $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ *and* $\alpha \in (0,1)$, *then*
$(x_1 \mathbb{I}_{S_1}, x_2 \mathbb{I}_{S_2}, ..., x_n \mathbb{I}_{S_n}) \succsim^* (y_1 \mathbb{I}_{S_1}, y_2 \mathbb{I}_{S_2}, ..., y_n \mathbb{I}_{S_n})$ *implies*
$(..., (\alpha x_i + (1-\alpha) z_i) \mathbb{I}_{S_i}, ...) \succsim^* (..., (\alpha y_i + (1-\alpha) z_i) \mathbb{I}_{S_i}, ...)$.

The beginning at the end

Proposition

Let $I = \{1, ..., N\}$ *and* $\succsim^*$ *be a binary relation on* $\mathcal{X}^*$. *Tfae:*

(i) $\succsim^*$ *satisfies SDEU, TRAN, SCOM, MONO, CONT and INDI;*

(ii) *there exist two families*

- $\{u_i\}_{i \in I}$ *of strictly increasing continuous functions* $u_i : \mathbb{R} \to \mathbb{R}$,
- $\{p_i\}_{i \in I}$ *of strictly positive probability measures* $p_i \in \Delta(S_i)$,

such that, given $\mathbf{X}, \mathbf{Y} \in \mathcal{X}^*$,

$$\mathbf{X} \succsim^* \mathbf{Y} \iff \sum_{i \in I} u_i^{-1} \mathbb{E}_{p_i}\left[u_i\left(X_i\right)\right] \geq \sum_{i \in I} u_i^{-1} \mathbb{E}_{p_i}\left[u_i\left(Y_i\right)\right].$$

The elements of $\{u_i\}_{i \in I}$ *are unique up to positive affine transformations, the elements of* $\{p_i\}_{i \in I}$ *are unique.*

Conclusions

- In decisions under uncertainty it is natural to associate different sources of risk with different criteria

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- In decisions under uncertainty it is natural to associate different sources of risk with different criteria
- Our approach reconciles the methods of classical decision theory under uncertainty with the spirit of multi criteria decision aiding
- It can be easily connected with current trends of decision making under ambiguity and dynamic decision problems