Feasible Optimization

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Basic ideas

- Usually a utility function, or, more in general, a decision model is elicited on the basis of some preference information supplied by the Decision Maker (DM)
- We propose to elicit in the same way a "possibility frontier" of the configurations in a complex system **S** (for example an enterprise). This is represented by a feasibility function which for any configuration says to which configurations it is possible to pass. The feasibility function is elicited from technical information supplied by one or more experts.
- Knowledge of the utility function and the feasibility function permits to find the most preferred configuration that can be attained from the current configuration of system **S**.
- The proposed methodology can be used for planning complex systems.

Plan

- Problem statement and basic concepts
- Feasibility function
- Didactic example
- Discussion on the concept of optimum within feasibility optimization
- Conclusions

Problem statement

- A set of alternatives A (actions, solutions, objects),
- m criteria from a consistent family $G = \{g_1, \ldots, g_m\} = \{1, \ldots, m\}$,
    - Ranking
    - Choice
    - Sorting

Aggregation means

- Aggregation using Multiple Attribute Utility (MAUT) functions (Keeney & Raiffa 1976):

$$U(a) = \sum_{j=1}^{m} u_j(g_j(a)),$$

- Aggregation using binary outranking relations S (Roy, Bouyssou, 1993):

$$aSb \Leftrightarrow a \text{ is at least as good as } b$$

Direct vs Indirect preference information

- Direct: The Decision Maker (DM) provides directly all parameters of the considered preference model,
- Indirect: The DM provides some preference information on a set of reference alternatives from which parameters of the model compatible with the preferences provided by her can be elicited through ordinal regression. (E. Jacquet-Lagreze and J. Siskos, 1982).

Basic elements of the problem

Let us consider a decision problem related to a selection of a future state in a complex system **S** for which:

- a set of criteria $G = \{g_1, \ldots, g_n\}$ represents the axis on which the performances of the considered system **S** are evaluated,
- **A** represents a set of alternative possible configurations of the system.

We suppose:

- $g_i : \mathbf{A} \rightarrow X_i, X_i \subseteq \mathbb{R}$ for all $g_i \in G$,
- $g_i(a) \geq g_i(b)$ if a is at least as good as b with respect to criterion g_i for all $g_i \in G$ and for all $a, b \in \mathbf{A}$.

Consider the set $\mathbf{X} = X_1 \times X_2 \times \ldots X_n$. Clearly $\mathbf{A} \subseteq \mathbf{X}$.

Preference relation and feasibility relation

On $\mathbf{X}$, two binary relations are defined:

- a weak preference relation $\succsim$, such that for all $a, b \in \mathbf{A}$, $a \succsim b$ means "a is at least as good as b",
- a feasibility relation $\trianglerighteq$, such that for all $a, b \in \mathbf{A}$, $a \trianglerighteq b$ means "it is possible to pass from configuration a to configuration b".

We suppose that both $\succsim$ and $\trianglerighteq$ are complete preorders, i.e. complete and transitive.

We suppose that:

- preference $\succsim$ on $\mathbf{X}$ is represented by a utility function

$$U : X_1 \times \ldots \times X_n \to [0,1]\,, \text{ such that } U(x_1,\ldots,x_n) = \sum_{j=1}^{n} u_j(x_j)$$

such that, for all $a, b \in \mathbf{A}$

$$a \succsim b \Leftrightarrow U(g_1(a),\ldots,g_n(a)) \geq U(g_1(b),\ldots,g_n(b))$$

- feasibility $\trianglerighteq$ on $\mathbf{X}$ is represented by a feasibility function

$$F : X_1 \times \ldots \times X_n \to [0,1]\,, \text{ such that } F(x_1,\ldots,x_n) = \sum_{j=1}^{n} f_j(x_j)$$

such that, for all $a, b \in \mathbf{A}$

$$a \trianglerighteq b \Leftrightarrow F(g_1(a),\ldots,g_n(a)) \geq F(g_1(b),\ldots,g_n(b)).$$

For the sake of simplicity, in the following we assume that $u_j(\cdot)$ and $f_j(\cdot)$ are piecewise linear non-decreasing functions of x_j, $j = 1,\ldots,n$.

Discovering utility and feasibility functions

Since $U(\cdot)$ and $F(\cdot)$ are supposed to be piecewise-linear functions, for each criterion g_j, the interval of evaluations $X_j = [\alpha_j, \beta_j]$ is divided in γ_j, $j \geq 1$, sub-intervals

$$[x_j^0, x_j^1], [x_j^1, x_j^2], \ldots, [x_j^{\gamma_j - 1}, x_j^{\gamma_j}]$$

where $\alpha_j = x_j^0$ and $\beta_j = x_j^{\gamma_j}$.

The marginal utility of an alternative a having evaluation on criterion j, $g_j(a) \in \left[x_j^k, x_j^{k+1}\right]$, is obtained by interpolation as follows

$$u_j(g_j(a)) = u_j(x_j^k) + \frac{g_j(a) - x_j^k}{x_j^{k+1} - x_j^k}\left(u_j\left(x_j^{k+1}\right) - u_j\left(x_j^k\right)\right).$$

Analogously, the marginal feasibility of an alternative a having evaluation on criterion j, $g_j(a) \in \left[x_j^k, x_j^{k+1}\right]$, is obtained by interpolation as follows

$$f_j(g_j(a)) = f_j(x_j^k) + \frac{g_j(a) - x_j^k}{x_j^{k+1} - x_j^k}\left(f_j\left(x_j^{k+1}\right) - f_j\left(x_j^k\right)\right).$$

Optimization problem for obtaining the utility function

$$
\left.
\begin{array}{l}
\max \varepsilon, s.t. \\
U(a) \geq U(b) + \varepsilon, \ \text{ if } \ a \succ b, \\
U(a) = U(b), \ \text{ if } \ a \sim b, \\
u_j(x_j^k) \leq u_j(x_j^{k+1}), k = 0, \ldots, \gamma_j - 1, \forall j = 1, \ldots, n, \\
u_j(x_j^0) = 0, \forall j = 1, \ldots, n, \\
\sum_{j=1}^{n} u_j(x_j^{\gamma_j}) = 1
\end{array}
\right\}
$$

Optimization problem for obtaining the feasibility function

$$\left.\begin{array}{l}
\max \varepsilon, s.t. \\
F(a) \geq F(b) + \varepsilon, \quad \text{if } a \trianglerighteq b \text{ i.e. it is possible passing from } a \text{ to } b, \\
f_j(x_j^k) \leq f_j(x_j^{k+1}), k = 0, \ldots, \gamma_j - 1, \forall j = 1, \ldots, n, \\
f_j(x_j^0) = 0, \forall j = 1, \ldots, n, \\
\sum_{j=1}^{n} f_j(x_j^{\gamma_j}) = 1
\end{array}\right\}$$

Feasible optimization

Let us suppose that the system $\mathbf{S}$ *has the current configuration* $(z_1, \dots, z_n)$*; which is the optimal configuration that could be reached from* $(z_1, \dots, z_n)$*?*

One needs to solve the following optimization problem

$$\max U(x_1, \dots, x_n), s.t.$$

$$F(z_1, \dots, z_n) \geq F(x_1, \dots, x_n)$$

$$(x_1, \dots, x_n) \in \mathbf{A}.$$

Didactic Example

(a) Evaluations of the eight alternatives on the considered criteria

	g_1	g_2	g_3
a	5	3	6
b	4	4	9
c	7	2	1
d	4	8	6
e	5	4	0
f	4	1	10
g	2	7	8
h	7	0	7

(b) Breakpoints

g_1	g_2	g_3
2	0	0
3.67	2	3.33
5.33	4	6.66
7	6	10
	8	

Getting the utility function...

Considering the following preference information:

- b is preferred to g,
- a is preferred to h,
- f is preferred to c.

(c) Marginal Utilities at break-points

g_1	$u_1(g_1(\cdot))$	g_2	$u_2(g_2(\cdot))$	g_3	$u_3(g_3(\cdot))$
2	0	0	0	0	0
3.67	0.1609	2	0.3524	3.33	0.001
5.33	0.1609	4	0.3534	6.66	0.002
7	0.1629	6	0.3544	10	0.4818
		8	0.3554		

(d) Utility of the considered alternatives

	g_1	g_2	g_3	$U(\cdot)$
a	5	3	6	0.5164
b	4	4	9	0.8536
c	7	2	1	0.5156
d	4	8	6	0.5183
e	5	4	0	0.5161
f	4	1	10	0.8191
g	2	7	8	0.5493
h	7	0	7	0.2137

Getting the feasibility function...

Considering the following preference information:

- It is possible passing from the configuration b to the configuration a,
- It is possible passing from the configuration d to the configuration g,
- It is possible passing from the configuration f to the configuration c.

(e) Marginal Utilities at break-points

g_1	$f_1(g_1(\cdot))$	g_2	$f_2(g_2(\cdot))$	g_3	$f_3(g_3(\cdot))$
2	0	0	0	0	0
3.67	0.4408	2	0.001	3.33	0.001
5.33	0.4418	4	0.002	6.66	0.2378
7	0.4428	6	0.003	10	0.5532
		8	0.004		

(f) Feasibility of the considered alternatives

	g_1	g_2	g_3	$F(\cdot)$
a	5	3	6	0.6339
b	4	4	9	0.9017
c	7	2	1	0.4441
d	4	8	6	0.6358
e	5	4	0	0.4436
f	4	1	10	0.9946
g	2	7	8	0.3678
h	7	0	7	0.7127

The feasible optimization

Table: Maximal Utility

Alternative	g_1	g_2	g_3	$U(\cdot)$	$F(\cdot)$	Alternative	g_1	g_2	g_3	$U(\cdot)$
a	5	3	6	0.5164	0.6339	a^*	2.2908	8	10	0.8652
b	4	4	9	0.8536	0.9017	b^*	3.3054	8	10	0.9629
c	7	2	1	0.5156	0.4441	c^*	1.99	2	8.834	0.6667
d	4	8	6	0.5183	0.6358	d^*	2.2990	8	10	0.8659
e	5	4	0	0.5161	0.4436	e^*	2	2	8.82	0.6659
f	4	1	10	0.8191	0.9946	f^*	3.6574	8	10	0.9968
g	2	7	8	0.5493	0.3678	g^*	3.3746	8	3.3297	0.4888
h	7	0	7	0.2137	0.7127	h^*	2.5891	8	10	0.8939

- For example, it is possible passing from the configuration $a \equiv (5, 3, 6)$ to the configuration $a^* \equiv (2.2908, 8, 10)$ having an increment of $0.8652 - 0.5164 = 0.3488$.

Postulates of the optimum and of reality of the first order

"**Postulate of the optimum:** In situations likely to involve decision making, there will be at least one optimal decision, namely a decision for which it is possible (on the condition that we have adequate time and resources) to establish objectively that a clearly better decision does not exist. It should be possible to do this while remaining neutral in terms of the decision-making process itself." (Bernard Roy, 1985 & 1996).

"**Postulate of reality of the first order:** The principal aspects of reality (an individual's preferences, the borderline between possible and impossible, the consequences of an action) on which decision aid is based relate to objects of knowledge that can be seen as both given (existing outside any modelling thereof) and sufficiently stable (in the face of duration, diversity of actors, discourse held, observations made) to legitimate reference to the exact state or the precise value (which can be of either a certain or a stochastic nature) of those specific characteristics judged significant of one aspect of reality." (Bernard Roy, 1993).

Feasible optimization and the two postulates

Postulate of the optimum: feasible optimization is not based on the idea that there will be one objective optimal decision. The whole process is only a basis to start a discussion with all actors participating to the decision.

Postulate of reality of the first order: The individual's preferences, the borderline between possible and impossible, and the consequences of an action are not supposed given and sufficiently stable. The ordinal regression to build the utility function and the feasibility function have to be used to co-construct a decision model with the actors participating to the decision process.

The constructive approach of feasible optimization

“The concepts, models, procedures and results are here seen as suitable tools for developing convictions and allowing them to evolve, as well as for communicating with reference to the bases of these convictions. The goal is not to discover an existing truth, external to the actors involved in the process, but to construct a 'set of keys' which will open doors for the actors and allow them to proceed, to progress in accordance with their objectives and systems of value.” (Bernard Roy, 1993).

Conclusions

- We presented a model for defining objectives when planning complex systems.
- The methodology proposes to elicit a utility function U representing preferences on configurations of a complex system $\mathbf{S}$ and a feasibility function F representing possibility to pass from one configuration to another of $\mathbf{S}$.
- The knowledge of the utility function U and the feasibility function F permits to define the most preferred configuration that can be attained starting by the current situation of the system $\mathbf{S}$.
- We used ordinal regression to induce one utility function U and one feasibility function F.
- In future research we plan to use robust ordinal regression to define a set of utility functions, compatible with preference information, as well a set of feasibility functions, compatible with technical information.

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